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Analysis of the Bodleian baroque lute stringing chart

Lists of integers beside lute course numbers have been found recently by Andreas Schlegel and Matthew Spring in a 1670 manuscript. These appear amenable to much useful interpretation, and also raise a host of inferences and queries. It was good of them to make this available and the initial summary in Ref 1 has led to further discussion in Refs 2, 3.

Over the last year I have tried out several different approaches, including a general close reading, analyses in terms of diameters, mass per unit length, total mass, tensions, patterns in the numbers, labelling, and even price. The aim was to understand the original ancient thought and practice, an approach I have found crucial in other areas such as temperament, and detailed properties of strings and lutes. A chief concern was to retain the initial integers as long as possible in the analyses, rather than losing sight of them by immediate conversions into the modern measures listed above. It has been possible to set up a single list, from which one can assess significant variations, and which requires only a few calculations. I have also made a simple device that illuminates curious features of the integers and the theoretical deductions, and could produce similar lists for specifying strings. The ancient source may describe a stringing similar to some modern set-ups, but the detailed properties of strings remain a continuing interest, and decisive analyses are given for two related but confused areas.

<u>Preliminary Matters</u>

Ideally one should start with many questions such as: who wrote the manuscript and lists and when; are they original work; are the lists personally determined properties or labels for strings; were they recommended by other authorities such as teachers; were they private or shared with others; did anyone use the lists to select strings or were they personal curiosity for the manuscript's owner, similar to Talbot; were the lists sufficiently accurate or in need of changes; were they standard specifications for widely available or even commercially labelled strings, similar to modern diameters in envelopes, etc? Few details are available, but these matters are worth bearing in mind.

A serious problem we need to remember throughout this study is that there is no original statement that the integers represent a particular string property. This has some interesting differences and similarities with my recent work on fretting instructions, where the numbers clearly represent fret positions but one wants to establish their significance and use (Ref 4). For later work on surviving citterns one clearly had valuable real frets but wanted to deduce the tuning system, although details such as string length and fret height could be uncertain (Ref 5). The present study is a constant search for the quantitative significance of the numbers and their patterns. This is similar to more familiar 'reverse engineering' where one wants to know how an old surviving object was made.

Much of the technical analysis therefore concentrates on the feasibility of different properties. Candidates can be eliminated because their predictions are highly unrealistic, or contrary to ancient mental approaches, or impractical for old methods. A powerful method here is the type of integers, their scatter for a single string, and patterns in their variation between strings. Although this is mathematical, it strongly indicates the ancient practical method, and how they set up their stringing schedules, to use a modern term. This should become clearer in the following analysis. Practical constraints include the probable use of gut in 17th century France, the process of twisting, and the available methods of measurement.

The appearance of the lists may suggest some form of labelling and supply similar to modern use, which would be a large change from earlier indications of experience, trial and error. For detailed technical reasons a single exact numerical deduction has not been possible, but a reasonably narrow range of options can be defined. It is possible that a simple future piece of information could be the key to a complete solution. While there is no difficult science in setting up a string schedule, even today would anyone know what Pyramid 1016 meant, without an explanation. The present work is therefore presented with some trepidation. However, one can deduce many qualitative aspects of ancient stringing that are at least as important as a single quantitative example. In this area of research, science and a historical view can be more productive than familiarity with modern technology.

<u>Methods and Data</u>

Firstly, it is useful to list several standard alternative relations for the vibration frequency f of a string:

$$f = \left(\sqrt{T/\mu} \right) / 2l \;=\; \left(\sqrt{T/\pi\rho} \right) / d\, l \;=\; \left(\sqrt{T/M\, l} \right) / 2 \;=\; \left(\sqrt{\sigma/\rho} \right) / 2l$$

In the first expression l is the sounding length of string, μ is the mass per unit length and T is the tension. This is the simplest in wave theory, and interestingly the most likely explanation for the ancient integers. It is often named after Mersenne, from sonometer experiments before Newton's theoretical mechanics. The second alternative just expresses μ in terms of the density ρ and the string diameter d, as $\frac{1}{4}\pi d^2\rho$. Musicians and makers may be most familiar with this form, named after Taylor. The third relation uses the total mass $M = \mu l$. The fourth form uses the stress σ from $T/\frac{1}{4}\pi d^2\sigma$, which removes consideration of any particular T and d. Ref 6 has more detail on the use of these relations for lutes.

The two lists of integers in Ref 1 can be written side by side as:

Course		Lists in MS		$n = 2.6/f^2$	$n = 5/f$
1	f′	5	5	2.6	**5**
2	d′	6	6	3.8	**6**
3	a	8	8	6.6	**8**
oct 6	a	(?)	(?)	6.6	**8**
oct 7	g	9. F	9,10 .i.	8.2	**9**
4	f	11,11½,12	11,12	10.4	10
oct 8	f	(?)	(?)	10.4	
oct 9	e	12,12½,13	13,14 .i.	11.9	10.7
5	d	15,15½,16	15,16	15.0	12
oct 10	d	(?)	(?)	15.0	
oct 11	c	18,19,20	18,19 .i. (?) ,20	18.5	
6	A	24,25	24	26.6	
7	G	30,32	32	32.7	
8	F	38,40	42,43	41.6	20
9	E	46,47	49,50	47.4	
10	D	58,60	66?	59.9	
11	C	73,75	82	74.0	26.6

The first list is from the original neat table, and the second list from the less neat original, with both lists in the same hand. The numbers increase as one reads across the lists, which should help discussion. The open string notes are provided clearly by the drawing of a baroque lute fingerboard, in the neat hand. This is worth stating because without it there would be room for uncertainty. While the bass strings have nominal pitches of G, F, E, D, C, music in certain keys would require tunings to $G^\#$, $F^\#$, E^b, and the C to $C^\#$, B, B^b. Also, I have taken the 'oct' entries from beside their basses and arranged them in order of pitch. This helps acoustic analysis but should not hinder their clear value of about a quarter the bass integers. Where no octave strings are specified, for courses 6, 8, 10, there is an inference they may be the same as unison strings on 3, 4, 5. This will be my only table, and the remaining two columns contain much of the analysis. There are many different interrelated aspects, so a simple orderly presentation is not possible, and there will be a constant mixture. Apparently abstract theoretical analysis sometimes leads to firm practical conclusions.

Initial Inspection

One can start by noting that the integers increase steadily with decreasing pitch down the list, indicating thicker strings, and with no overlap in the scatter on adjacent pitches. For the basses 6 down to 11 there are generally two integers, with little spread in each list and somewhat more between lists. The largest relative spread is 8 from 58 on course 10, which is plus or minus 4 in 62 or about 6%. (The ? refers to an unclear second digit, but course 11 has similar scatter.)

The next group, from the octave g on 7 down to the octave c on 11, often has three numbers, from a ½ inserted between integers, or three consecutive integers for the larger numbers on octave 11. The largest spread here is 2 from 12 on octave 9, which is about 8%, similar to the basses. The numbers may seem messy but are reasonably tight for defining some measure of string thickness. The similar proportional scatter on the different strings is useful for describing some physical process. The intermediate ½ might be an estimated measurement, or a preference for a slightly bigger 11 or smaller 12 in supplied strings. (There are also extra symbols on the octaves such as an F, and an .I. with the dots higher, which seems familiar but untraceable on the wordbound web.)

The last group is course 3, and a presumed octave 6, up to course 1. This looks simple but is rather alarming, because each course is defined by the same single integer in each list. There is no measure of uncertainty, and it might appear that each quantity is completely accurate, as taken in Refs 1,3. Initially, I would suggest an effective scatter of ¼ to ½, in line with the lower strings. In specifying a string for course 1 it might have appeared that numbers 4 or 6 were in some sense too extreme. Also disturbingly, there is scatter on octave 7 so one would expect a similar 5 or 10% on course 3 and on a presumed octave 6, which are important A courses. Now we can examine the likely meaning of the integers.

Diameter of String

A description of string diameter can be ruled out. Taking the f-F octave from course 4 down to 8, the numbers increase from about 11 to 40. A fourfold variation of diameter, with a constant tension, would produce a two octave change of pitch. The whole range from C to f ′ has a pitch factor of 16/3 (2 octaves and a fourth) whereas the numbers have a factor of about 16, or 4 octaves in terms of diameters. This inspection of octaves is a neat way of studying possible relations and mechanisms, and the low d-D is similar to the low f-F, but some care is needed. The upper f ′-f octave has an integer ratio of only about 2, which might initially be seen as suitable for an octave. In relation to the lower courses, this can be explained by an increasing tension above about course 3, as seen from the above equations and examined below. For each of the lower octaves, a description in terms of diameter would require a large fourfold increase of tension towards the bass, which is clearly impossible.

Strand Number (see Appendix for further implications)

This is the description adopted in Ref 1. One can see that n identical strands of gut each of diameter d_s, when aligned parallel and collapsed into a single cylinder would produce a diameter $d = d_s\sqrt{n}$. This effective diameter is seen from the areas or masses with $(\frac{1}{4}\pi\rho)d^2 = (\frac{1}{4}\pi\rho)nd_s^2$, since the collapsed mass has the mass of n strands. The need for an effective diameter in Ref 1 arose from using the familiar Taylor equation. The diameter for each pitch was proportional to $\sqrt{n}$, with course 1 set at a typical modern value of 0.47mm. The above equations show that with values for the open pitches, an open string length and density, then tensions of each string can be calculated as $\mathbf{T = \pi\rho n d_s^2 \mathit{l}^2 f^2}$. Ref 1 used a top f ′ pitch of 313Hz, or an a′ of 392Hz, and a gut density of 1.3 gm/cc, but for only the second, less neat, list of larger integers. The unspecified *l* is around 70cm, by reverse calculation. These are all reasonable assumptions, but the original ancient values could have been quite different.

My first impression from all the integers for the lowest two groups of strings from C up to g was an essentially constant tension, simply on account of the approximately fourfold increase of n for pitches an octave lower. This is a clear trend for all notes, although not completely exact, and with intermediate pitches fitting appropriately, as seen below. In the third group from 'a' to f ′ the strong increase in n, such as 5 rather than a 3 corresponding to 12 for the lower f, indicated a steady increase in tension for higher strings. The list of calculated tensions in Ref 1 also has these trends, but technically minded readers would not assign much initial importance to the relatively small variations of tension in the lower two groups.

The initial notion of strands has many problems such as the ½, and the large number of 5 for a thin top string normally having fewer strands (see also Refs 1,2). A more devastating problem is that real strings would be twisted. A probable graded increase of twist for lower strings would hinder calculation of relative string diameters, and assumptions of twist from modern strings would not be reliable. An interesting aspect of strands might be an implication of a single material such as gut, without the use of extra dense materials, but this would not be certain.

All these factors indicated that the attractive features of the strand calculations would be retained if the integers were measures of mass per unit length μ, or total mass M of each open string. The dependency of pitch f on $1/\sqrt{n}$ for a constant tension would be retained, and similarly the behaviour of the top three courses. As a physicist, I saw that this description is formally equivalent to the strand idea since μ is just the same as $\frac{1}{4}\pi\rho d^2$ or $\frac{1}{4}\pi\rho n d_s^2$ for the parallel strands, and most importantly there is no complication from any twist. (A while later I saw Robert Venning's letter in Ref 2 also suggesting weight of a specified length of string, and in Ref 3 Richard Corran gave lists of diameters and tensions mirroring those in Ref 1.)

Mass of String

In the above discussion, the integers could be direct measures of the relative masses of the strings. Then the relative values of tension for the 14 different strings plus 3 assumed octaves could be found. Absolute mass per length would require at least one real mass of a known length, or equivalently one real diameter and an effective density. Absolute tensions, in kg or Newtons, additionally need values for the original pitch standard and string length. These four extra values produce a single scaling factor. The type of treatment in Ref 1 could lead to many long lists, often just to change the scaling factor, but tending to obscure the original integers.

The range of tensions in Ref 1 was 43N on course 1 to a low 20½N on course 6. These tensions would be quite extreme compared with typical modern uses of 40 to 25N. No variation of the extra scaling values could narrow the relative range. It might seem that the tight bracketing somehow validates the assumed values, and perhaps a little juggling was required to accommodate both extremes. However, if one recognizes the effect of scatter, uncertainty, and the other neat list of integers, then alternative levels of tension would be possible. The integers may also contain information or inferences for an absolute mass per unit length, or diameter. In principle, this could lead to the absolute tensions in ancient use, but as seen below this is far from certain.

For all the above reasons it seemed useful to set up a more general expression for the variation of relative tension with the raw integers n that are potentially proportional to mass per length μ. This would keep the valuable original data in constant view rather than cover it with modern assumptions. Effects from variations of the integers could be seen directly, and a simple step would add the various scaling values.

There are two sections: one for the lower strings from g on octave 7 down to C on course 11, and one for the g up to f ′ on course 1. The main low section can be reasonably well described by a relative pitch f equal to **$\sqrt{(2.6/n)}$** for a constant tension T. The relative pitches take the course 1 f ′as a standard and using perfect ratios run down as:

f ′	d ′	a	g	f	e	d	c	A	G	F	E	D	C
1	5/6	5/8	9/16	1/2	15/32	5/12	3/8	5/16	9/32	1/4	15/16	5/24	3/16

These perfect ratios are sufficient, with no need for tempering. (Interestingly, the top six courses f ′ to A could be spanned by perfect intervals since 5/4 times 4/3 times 6/5 is a perfect octave, but tempering would be needed to accommodate the fretted notes. For vieil ton six ‘perfect courses’ would be a syntonic comma less than two octaves.) The values of n calculated from **$n = 2.6/f^2$** are listed in the above table. From g down to C there is good agreement, with values lying within the range of the original lists. Slightly low n are found for g and f, but rather higher n for A and G. The relative error on g is about 0.8 in 9, or 9%, but here one can see an increasing departure for the higher courses, as explained later. The error on A is about 1.6 in 25 or 6%. The extreme deviations of 1.8 on g and 2.6 on A are twice as large. The factor of **2.6** was reached by simple inspection, with 2.5 improving A and 2.7 improving g. An elaborate best fit is not necessary or appropriate. These results can be used to calculate effects of the small departures of the original integers, and to find constant tensions for different scaling values. This will be seen after treating the upper section.

The section above g can be expressed exactly by **$n = 5/f$**, as shown on the list. This is simple, but extremely peculiar. As seen in the list of frequency ratios, the integers n follow exactly the perfect intervals between the notes f ′, d ′, a, and even g, that would hold for diameters at a fixed tension. For masses, however, a pitch varying as $\sqrt{(T/\mu)}$ and also as $1/\mu$ means that the tension varies formally as $1/\mu$ or 1/n, which is a strong increase for smaller n. The integers begin to depart from the lower constant tension region near g or f. The matching point is at n = 9.6, from $2.6/n = 25/n^2$.

For ‘a’ the actual n is about 20% above 3.8; for d ′ the increase is 60%; and by f ′ the increase is a large 90%, almost a doubling.

The tension at n = 5 is increased from the constant value at n = 9.6 by a factor of 1.92 (or 9.6/5). At n = 6 the factor is 1.6, and at n = 8 the factor is 1.2. This steeply rising tension would be very sensitive to realistic uncertainties in the original integers for the highest two courses. Without courses 1 and 2, the tension might seem reasonably constant for nine courses, but the increase really starts on courses 3 and 4.

This may seem abstruse but there are several immediate practical uses. The range of tensions, from a lower constant for C to g, then rising up to a tension 1.92 times greater on course 1 gives the following examples of ‘lower constant & top course tension’:

18 & 35N , 20 & 38, 22 & 42, 24 & 46, 26 & 50, 28 & 54N

Modern experience might find 18 and 50N most extreme, leaving a very narrow useful central range. If scatter lowered the integer on course 1 to 4.5, then the factor 1.92 reduces to 1.75, and the range widens a little to

20 & 35 N , 22 & 38, 24 & 42, 26 & 45, 28 & 49N

An example of a practical absolute lower tension can be calculated from

$$T = 4\mu\, l^2 f^2 = 4M\, l\, f^2 = \pi\rho\, d^2 l^2 f^2$$

Although the analysis centres on mass per length, diameters will be more familiar for illustration. This requires just a single absolute value of d, and for our low F a reasonable modern value is 1.4mm, also close to Ref 1. With 1.3gm/cc, 70cm and 78Hz (312/4), a tension of **24N** is found, and this applies for all pitches below g. For courses 3 and 2 the tensions are 29, 38N (from the factors 1.2, 1.6), and for course 1 the tension is 46N, or a lower 42N from 4.5.

To calculate the tension for any of the original integers, or perhaps an average, the tension for say integer 38 on F is simply lowered by 38/41.6, or a factor of 0.9135, from 24 to 22N. This applies to all other lower notes, and clearly also to relative tensions. The upper notes have no original alternatives, but the method used for reducing course 1 could be applied. This approach has many benefits over repetitions of the full tension expression. There can be no objection to using a special form since it is unlikely that much more ancient data will be found.

Absolute mass per unit length

The above example is a good practical representation of the original integers, but in addition to the scaling of ρ, l, f, a single real value of d, μ or M was needed. The integers clearly suggested a possible original ancient relation to measures of mass or weight in 17^{th} century France. For our F string above, the mass of a sounding length of 70cm would be $M = \frac{1}{4}\pi\rho d^2 l$ which is 1.4gram. For an integer of around 40 for F, an ancient unit of order 0.035gm (1.4/40) or 35mg would be needed. The apothecaries' or jewellers' grain first came to mind, and searches gave the sequence of units:

Livre = 489.5gramme (equivalent to about 1.097 imperial pound)
once = 1/16 livre = 30.59g; then a denier scruple = 1/24 once = 1.275g
grain = $1/24^2$ once = 53.11mg; and a prime = $1/24^3$ once = 2.213mg.

The string mass of 1.4gm would be 26.4 grain, but about 1scruple or 632prime, so the intermediate and more familiar grain seemed most appropriate. Ideally and scientifically we would like integers to refer to a sounding length, close to 70cm, because this would be a direct measure of the mass per length and hence diameter, close to a full specification. However, it would be very impractical for ancient workers to produce this description, for several reasons detailed below. The integers were unlikely to refer to a sounding length but to a greater supplied length convenient for fitting on a lute. In the example this length would be 40/26.4 x 70cm or 106cm. Such a length would be enough to fit a single string on a standard baroque lute, and perhaps some larger lutes or two strings for much smaller instruments. The same result holds for all the other strings and integers, by the same proportion of μ or d^2 to the integers, so that lists of calculations are unnecessary for analysis but could provide an illustration. (After the present work Ref 3 appeared, also proposing the grain.)

These exploratory calculations looked interesting but it was soon clear that they were unlikely to lead to a full description of string geometry and acoustics. However, with care it has been possible to tease out some ancient methods and thought. This will be useful, since there will be no single absolute final string schedule. In fact, the above analysis of the relative masses and tensions together with practical limits may be a better guide than any potentially variable ancient units. Accordingly, later sections will give more detail on the patterns of integers, and then suggest a practical weighing device that would not require absolute units.

If all 40grains of our F string were confined to the sounding length of 70cm, then mass per length would be increased by a factor 106/70 = 1.51, and diameter by 1.23. A constant tension as low as 20N would be increased to 30N. An F diameter of 1.4mm would be increased to 1.72mm, and a top course would have 58 or 52N, all of which are impractically large.
A longer string of 140cm would produce corresponding tensions of 15 and 29 or 26N, with a diameter of 1.22mm, and a light slack touch. One can see that mass per length, and hence supplied string length, would need to be accurate to about 10% to be of much quantitative use.

Length of Strings

Having found the grain, the next step was a search for ancient measures of length. Although the manuscript contained no numbers that might relate to length, if some standard unit or method of measuring were in our region of interest it might combine usefully with the grain. It might seem that an initial string of 106cm indicated above was very close to 1metre, and that some ancient forerunner would give an immediate plausible conclusion. However, there appears to have been no such unit. More alarmingly all French units of measurement before their 1789 revolution were extremely chaotic, with thousands of systems, and large variations by factors up to 2, between localities and over time. This was particularly marked for units of length, but also occurred with mass, and resulted from the types of trade, local concerns, fraud etc. (Just before the revolution the situation in France was so bad that steps were taken to standardize units, and much later led to their gramme, metre, etc, and finally the Systeme Internationale d'Unities.) The smaller ancient units of length have a similar form to the mass units:

Toise = 6 pied du roi = 1.949metre (corresponding with a fathom of 6 feet or 2 yards)
pied du roi = 32.48cm (1.066 imperial feet); pouce = 1/12 pied = 27.07mm (1.066in)
ligne = $1/12^2$ pied = 2.256mm; and a point = $1/12^3$ pied = 0.188mm.

The 'king's foot' and related units were reputedly consistent for 800years, but mostly used by learned upper classes while trades, including lutes and strings, used many variable units.
The toise was used for land as well as sea, and measured the distance between the fingertips of a man's outstretched arms. The yard derived from the nose tip to an outstretched fingertip, perhaps nearer 30in or 76cm, and still often used by older people today.

A unit of greater interest may be the ell, related to the Latin ulna, the length of the arm from the elbow (bow of the ell) to the fingertips, and about 18 to 22inches. This derived from the previous cubit, and many variations centred on a 'double ell'. The range can be seen in an ascending list: German about 23inch, 58cm; Swedish aln of 2fot, 59cm; Danish 25inch, 63.5cm; Flemish 27in, 69cm; Polish 31in, 79cm; Scottish 37in, 94cm (Gie 'im an inch, an h'ell tak an ell); English 45in, 114cm; French aune 54in, 137cm.

Woven materials, including textiles, tapestries and sail canvas, were traded in ells, often with different units for buying and selling. Large quantities of string and rope were usually sold by weight, but there are several early written relations between musical strings and haberdashery (eg gansars). The early procedure of measuring out cloth along an arm clearly gave rise to the ell unit. This might also have been used for short lengths of string, as for a lute.

The large range of ells should be a warning against rash conclusions, but a lower limit of about 80cm for our lute strings (70 + 10 for tying on), might rule out the small ells and leave only the large 94, 114, 137cm types. However, the large French aune of 137cm would lead to an F diameter of 1.23mm, with light slack strings as above. Taking the Scottish and English units as estimates of variation rather than standards, gives an average of 104cm with our typical F diameter of 1.4mm. An average of all three ells would be close to 114cm and a diameter of 1.3mm. This is the desired degree of accuracy but there can be no certainty in the assumptions. Alternative measures could be 3pied or ½toise, but probably not for cloth and string, or any arbitrary length that became a convenient standard. A simple practical method is to take the end of a string between forefinger and thumb, then loop behind the elbow, and bring up to the fingertips. This is how we still coil up rope, and now electrical cables. My measure is 39inches or 99cm, and wearing a jacket 41in or our recurring 104cm, which is also a simple 1½ times our 70cm sounding length.

Practical masses, lengths and bundles

The mass units now need reviewing. The livre was based on the weight of a 1/70th cubic pied of water, but did not have the stability of the pied. Lengths had many human measures, but there would not be many convenient units for mass. So the basic units themselves became variable, but without record. Some idea of the probable variation might be inferred from the better-regulated English systems. Here troy weight, avoirdupois and the apothecaries' system all have a grain of 64.799mg, but with differences in the larger units. However, the tower system had 45.5mg, and a wheat grain was 48.6mg or ¾ of the 64.8mg barley grains.

The nominal French unit of 53mg may have had variations of order 10mg or 20%, and perhaps more. This would affect all the above estimates, but some initial yardstick was needed above for assessing absolute masses and lengths. An interesting limit is that a grain of low mass might not be able to define the mass of any strings in the lists. Using our typical F string of diameter 1.4mm, a length of 106cm was 40grains of 53.11mg, so the minimum size of grain needed to define a minimum useable length of about 80cm is 40mg (from 53x80/106). Thicker F strings would need a larger grain, of at least 52mg for a 1.6mm diameter, but only 22mg for 1.2mm. In strict terms, this may provide the only firm and practical conclusion one can draw from the idea of measurements in grains. Any ancient grain used must have been above about 40mg, plus a bit more to account for a highest integer in the scatter. A grain of exactly 53mg or the many hints of a 106cm string should not be taken too seriously. Just two numbers reliable to maybe less than 20% cannot be expected to give precise quantitative conclusions on absolute string masses, diameters and tensions. It is important, however, that they do not contradict, and can even support, the relative values of mass, or diameter, and tension found from the integers, but they cannot add much. (If anyone knows a specialist in ancient metrology, it would be safer to ask about units for a likely date and locality, before explaining the application.)

There are further interesting qualitative aspects of string length and mass. Firstly, the relatively smooth variation of the integers with pitch is consistent with the assumption of a fixed common length for all the courses. There are ancient references to early strings in long bundles (Ref 7), or later in hanks that are flattened coils of many long strings bound at the centre (Ref 8), and finally coils of separate strings in boxes. The coils seem to be sufficient for a single string, close to modern practice and perhaps the present case. The strings in hanks were sufficient for two lute strings, and thinner longer ones for higher courses may have provided four strings.

If the old strings in the manuscript were in two distinct lengths, one would expect a large jump in the integers. Alternatively, use of the actual integers could lead to much reduced diameters and tensions for the higher strings, which would be a greater surprise than the present large values. The opposite case of much longer bass strings, perhaps for other lutes, would lead to thinner basses and lower tensions than the calculated constant level, maybe continuing below the slightly low values found on courses 6 and 7. Moderate random variations of length are unlikely, because this would wipe out any significance of the small variations of integer between courses for describing string gauges. A gradual variation of length between courses seems unlikely but it could completely invalidate the simple calculations of mass per length and relative tension. This would be similar to an effect of twisting on diameters discussed above.

Lastly, if the integers refer to the mass of several similar strings, as for bundles etc, then larger units such as the scruple for perhaps 24 strings could be involved. The small variations of integer in the lists could make this an unreliable guide to finding a precise diameter needed for a single string. We are not justified in fiddling with variable numbers of strings in bundles. However, one can see that certain masses of scruple and grain for given integers would be inconsistent with some sizes of bundle. Practical methods are best continued after some further inspection of the integers. This leads to suggestions of how the ancients set up the lists of integers, and decided what strings to use, or in modern terms how they set up their stringing schedules.

Origin of the ancient integers

A physicist may look for a fourfold pattern, and then demonstrate it with the n function, but it is also seen that almost nowhere is there an exact pair of such integers. The only case is 15 for d and 60 for D. Everywhere else the integers are close, and the pattern lies within the scattered groups. The closest examples are 11, 43 for f-F; 12, 47,49 for e-E; 18,73, 19,75, 20,82 for c-C.

The e-E looks perverse since 48 is the only number missing in the group 46 to 50, but then it is seen from c-C how one group may cross but not exactly match the other. This shows that the integers were not derived from any strict mathematical use of formulae such as $f = (\sqrt{(T/\mu)})/2l$, as found in Mersenne's writings 30 years before the manuscript of 1670. This does not rule out some initial rough fourfold guide.

The same conclusion also follows from the difficulty ancient workers would have in finding integers theoretically for intermediate pitches. If we start from an exact f-F octave with 11 and 44, then the only fundamental way of finding n for other notes is from frequency ratios, so that d for example would have $11(6/5)^2 = 396/25 = 15.84$. This is numerically close to 16 in the lists, but one can be sure the ancients never made such calculations. Ancient fretting schemes devised by leading mathematicians generally used ratios for successive geometric constructions (Ref 4), so it is inconceivable that practical string suppliers or fitters carried out long multiplications and then rounded to the nearest integer. The other notes of interest are

g with $11(8/9)^2 = 704/81 = 8.69$, close to the listed 9
e with $11(16/15)^2 = 2816/225 = 12.52$, close to the listed 12, 12½, 13
c with $11(16/9)^2 = 176/9 = 19.55$, close to the listed 20
A with $11(8/5)^2 = 704/25 = 28.16$, higher than the low 25
G with $11(16/9)^2 = 2816/81 =34.76$, higher than the low 32
F with 44 slightly lower than 43; E with 50.06, close to 50
D with 63.36 close to 60 or 66; C with 78.2, close to 75 or 82.

The same behaviour occurs if different reference notes or alternative natural frequency ratios are used. The d-D octave is also a good base since it is the lowest two octave span. It might be said that the ancient workers approximated the (inverse) square function $n = 2.6/f^{\,2}$ in terms of integer masses derived from experimental acoustics, similar to locating the midpoint of a string by searching for the node of the second harmonic. Some practical conclusions are of interest before further details on integers for octave strings.

Practical aspects of string selection

From the preceding analysis it follows that the neat looking pairs of integers must stem from a largely experimental basis. This is not hard to appreciate since our earliest, and technically most useful, source on strings (Ref 7) advised setting the lowest three of the six courses, thereby preserving them as a stable reference for choosing the less durable upper courses. Players would compare look and feel in selecting new strings, and the experience of experts would be useful or essential. The process would have started with the earlier four courses, then extended to five, six, and seven with a low D course. The later addition of four bass courses would have presented little further problem in string performance, as explained in Ref 9, but one can see that with so many strings some guidance would be useful in selecting masses or gauges. This highlights a remarkable aspect of the original integers in specifying a smooth increase for all the six closely spaced basses A to C. Without this source we would not have known that this detailed variation was required and achieved by the original workers and players. Using roughly the same gauge on some courses would have saved considerable effort. Modern usage has adopted graded gauges for the sound and feel, which is easy with modern strings and formulae, and the ancient source now supports this. This is worth stating since it is easy to regret the absence of absolute masses etc, and forget straightforward positive information.

Many detailed scenarios for specifying strings can be envisaged, but one aspect has technical significance. If the integers represent the mass of a length of string suitable for fitting, then this could involve a laborious procedure. It has been shown above that masses were initially experimental, and we can consider the first time weighing was used. Imagine a length of string has been weighed, a little trimmed off, then fitted and found musically suitable. This provides, by experience and chance, an integer on the list, and maybe the second integer or a ½. If the string is not adequate, or could be improved, another string of equal length but different mass could be tried, until one was satisfied. In order to replace strings one needed to find an identical correctly weighed length of string. This might be much easier if strings were available from a group of players or commercially. The remarks above on ancient workers not using long calculations also leads to the earlier conclusion that the integers apply to strings longer than the sounding length. Defining masses for a sounding length, and making replacements, without using proportional calculations, would require many bizarre sequences of cutting up strings. This explanation seemed too lengthy to give above.

<u>Details of integers and octave strings</u>

The pattern of integers n led to an approximately constant tension region from course 11 bass C up to octave 7 g. This was anticipated from a characteristic fourfold change of n between notes an octave apart, with other notes suitably intermediate. This led above to the detailed function $\mathbf{f = \sqrt{(2.6/n)}}$, or $\mathbf{n = 2.6/f^2}$. From the lists, in strict descending order of pitch, it was seen for g and above that the original integers were progressively higher than the function. This trend may begin for f, with a lowest real 11 relative to the function's 10.4. The lower octave strings e, d, c would have tensions close to their basses, within the scatter of integers. In contrast, the octave strings f, g and especially 'a' in the upper region, would have tensions progressively greater than their ideal basses, by the factors 1.1, 1.16, 1.21 (from ratios 11 to 12/10.4; 9 to 10/8.2; 8/6.6). This indicates that, at some stage, octave strings were identified and chosen from a predetermined list of masses appropriate to pitches, rather than matched by trial to their basses. The actual tension differences could be even greater, since it was seen above that basses A, G were a little lighter than the surrounding sequences. From the lists, the ratio of integers for the octave with A is 3.06 (from 24 to 25/ 8), so the octave's tension is 1.31 (from 4/3.06) times the bass tension. For g the ratio is 3.32 (from 30 to 32/ 9 to 10), so the tension is 1.2 times the bass tension.

A method of fourfold patterns can also be used for modern strings, perhaps when planning uses for some strings one has in stock. A neat addition is to reduce the tension of lower strings by slipping the fourfold mass, or twofold diameter, to a note just below the octave. Such detail is seen in the ancient lists for g-F, f-E, where the upper strings used as octaves were shown to be tauter than the lower constant tension for G and F. The effect is less for e-D, and absent for d-C, where d-D and c-C give good equal tension octaves. These ancient cases, however, would result from experiment rather than theoretical guides that we can use today.

These observations lead to an intriguing ancient possibility, and also provide an explanation of the mysterious 'F' symbol next to 9 for the octave g in the neat list. This might have been a later suggestion for an alternative improved octave f with the F bass. This could also infer use of the 8 from 'a' for a better g, the 6 from d ′ for the octave 'a', and 11,12 from f for the e. These alternative fourfold relations are better than for the listed larger integers, as used in the initial analysis, and they would produce equal tension octaves rather than the unusual tauter octaves:

d ′: 6x4 = 24.	A: 24,25	theor 27
a: 8x4 =32.	G: 30,32	33
g: 9,10x4 = 36,40	F: 38,43	42
f: 11,12x4 = 44,48	E: 46,50	47

The last column is the function $n = 2.6/f^2$ for comparison. This observation indicates that the ancients were actively trying to improve their specifications from an initial list of strings based on pitches, as already obtained from weighings or labelled items.

Differences between the two lists

Looking in detail, the less neat list has integers which are slightly greater than the neat list, e g 10 from 9 on g, 14 from 13 on e, 43 from 40 on F, 50 from 47 on E, 66 from 60 on D, 82 from 75 on C, all about 10% increases. The other notes f, d, c, G are unchanged, and the octaves e, f, g, 'a' were discussed above, but most noticeably A is reduced from 25 to 24. The anomalously low A and G basses may also have significance for the ancient approach, and several features can be examined.

It was seen that all the lower integers of the neat list can be fitted closely to $2.5/f^2$. In contrast, most of the less neat higher integers are best fitted by a factor of 2.7, but the low G and lower A can only be roughly accommodated by a factor of 2.6. Overall, the less neat list gives slightly heavier tauter strings below course 3.

Maybe the neat list was a more desirable earlier standard, and the 'rough' list is a heavier attempt, perhaps later by the manuscript's owner. Although the neat list is on the right side of the page it may have been at the top of a page with the full width fingerboard above or below. If the 'F' on the neat list is a later improvement the 'rough' list may have preceded the neat one. Some aspects below may suggest this, or both lists may have been used together. The exact situation is not crucial here.

Almost all integers above course 8 are common to both lists. This may suggest that one list was adapted from the other by adding and changing some integers. These new integers are the added 10 for g, and 14 instead of 12 for e, both octave strings, and might even be small estimated changes in order to match heavier basses. In contrast, all the fourteen different integers on the lowest four basses would need to involve new weighings and testing, since it is unlikely the ancients had theory or other methods to make these extrapolations. In this process the A and G courses may have been overlooked, or initially thought accurate enough, or thicker strings were not available. The fact that one list, the neat one, has a smooth behaviour, indicates there was no unusual technical purpose behind the low tensions for A and G on the 'rough' list.

Another notable feature is that while the A is normally considered a bass, course 6 is also at the centre of the 11 courses and might be some initial reference. A process of string selection dating back to six course lutes may have left a trace of a break before the later addition of five lower courses.

Just above the curious A integers one sees that the relation between the A and c strings is different from all other notes below course 3. The general musical interval down the list of pitches is a tone, and this is reflected in a gradually increasing separation between integers. The only exceptions are the two F-E semitones, and the larger c-A of 3 semitones, which occurs just below small integer steps from f ′ down to d. The next step from d to c is 3 or 4, and yet from c to A is only 4, whereas one might expect at least 6, and even 8. This might indicate that some rough guide or expectation had not been modified for the larger interval. Integers of 27 or 28 may have improved A. The problem is less for G but 34 or 35 may be better.

Another remarkable feature is that the main integer for A is 24, and this is the number of grains in a 'denier scruple'. Is this just coincidence, or is there further evidence of this unit, and are there other numerical patterns in the lists not necessarily related to the acoustics, but serving as useful guides. Half a scruple is 12 grains, which appears for f and e in both lists. However, 2 scruples is 48 grains, and this is the single exception in the group 46 to 50. For 3 scruples, 72 is missing below 73, for 1½ scruples 36 is missing but there is a 38, and for 2½ a 60 appears for D. This may not seem promising, but the closely grouped integers do indicate a high degree of precision. Integers such as 13, 25, 73 would require just a single extra grain on the weight side of scales. Slightly low integers such as 11, 47, 58 initially seem a problem, requiring many small weights, but a neat method could be an extra grain on the string side. This may seem compelling until one notices that integers with the very different base of 5, such as 10, 15, 20, 25 etc are also very common. The ‘24’ at the central point, with the new basses below, remains intriguing.

Possible reasons for the intermediate ½ and 19, and the single numbers for the upper three or four courses in the two lists were discussed above. Aside from these numbers, each list generally has two very closely spaced integers, except for four single numbers on the ‘rough’, maybe less certain later list. The scatter within each list is usually only 1 on the smaller integers, and on the larger integers at most 2, which is relatively lower. For the middle courses this scatter, of order 10%, is comparable with differences between the lists and the accuracy of the fitted functions. For the lower courses the scatter of 2 is relatively much smaller, of order 2%, and much smaller than the 10% difference between the two lists. The difference within a pair of integers on the same list therefore has little significance for the acoustics. There might be a repeated measurement on a string, which could provide confidence. In summary, as a specification for alternative masses, the scattered integers would give a narrow range, satisfactory for middle courses, and negligible compared with the difference between lists for the lower courses. These abstract points will become clearer in the following practical study of the weighing process. The presence of many common integers in the two lists, indicating changes, by additional weighings, estimates, or even labelled items, is also significant.

A Simple Weighing Device and its Errors

The entire concept of integers representing masses of string depends on the practicality of weighing, perhaps for a single short length of the lightest gauge. Some may have doubted this (eg Ref 2), and certainly electronic kitchen scales designed for a few grammes may not even detect our lowest mass of about 0.25g. This is the small mass of a top f ′ string with an integer of 5 and length of 105cm, compared with 1.4g for our 70cm F string with an integer of 40, from 1.4x (5/40) x (105/70) = 0.25. My initial ‘feasibility test’ used a simple wire clothes hanger on a peg. A top f ′ string hung at an end produced a clear deflection of just over 1mm, and immediately proved the principle. The next improvement was a long wooden bar, a fine old boxwood metre rule, taped symmetrically to the wire horizontal. It is important to have the centre of gravity well below the peg to make a stable system. Further coils of various modern strings were tested, by hanging them on one end of the bar, to deflect the end relative to a vertical mm scale:

g ′	f ′	d ′	a	f	(d A)	F	D	F + f
2	3	4	6	8		31	40	39mm

The accuracy of all these deflections was about plus or minus ½ to 1mm. Weighing two strings confirms that the scale is linear, and many more strings and combinations could have been used. The device could be refined with an arm longer than 50cm, more sideways restraint to reduce parallax on the scale, a more precise pivot and a little friction. However, this short test demonstrates many effects in the lists. (The string lengths were all about 1metre but varied between 90 and 110cm and I did not want to cut them up, or trouble with correction factors.)

For deflections above about 6mm, the accuracy is adequate and one can discriminate between ‘a’ and f by a clear 2mm. The f-F octave shows a clear fourfold effect, but the d ′-D of two octaves is narrow due to low accuracy on the d ′ and a short D string.

For lower deflections, down to 2 or 3mm, the readings are little more than guesses, with an accuracy of around 50%, and one can barely distinguish the g ′ and f ′ strings. Weighing several strings in different combinations could give many economical systems of practical testing and quality control. For thin strings much more accurate weights could be estimated. For example, eight ‘5’ strings nominally weigh 40 units, so a real reading of say 36 would indicate a better average of 4.5. This division may seem too modern compared with just adding weights to a scale, but simpler methods of comparison seem possible.

All this has a strong parallel with the lists. The ancients may have used simple equal armed balance trays and a set of weights, but similar principles of accuracy would apply. British pharmacists used scruple and grain weights until decimalization in 1971, with 20grains of 64.8mg in a scruple, and a ½ grain weight. Thin brass or aluminium sheets were cut into tiny pieces. A balance with a reference weight sliding on a scale towards a pivot could cause more inaccuracy for thin strings. Conversely, hanging the thinnest strings at a variable point could be highly accurate, and easy with a device like the hanger.

Before these tests it was seen above that the simple integers 5, 6, 8 for courses 1 to 3 were most likely subject to large random and systematic errors. They look analogous to my 2, 3, 4, 6 above, but not quite so vague. In light of these tests, the likely error on the integers might be realistically increased, from the previous ¼ or ½, up to 1. (The situation is probably worse because all sorts of measuring device tend to be unreliable near their zero, for a variety of reasons. This could set a minimum weight, such as the ‘5’, and it is often better to set up another reference point.) A top course integer of 4 would reduce the tension to about 1.54 times the constant lower tension, allowing a greater practical range of lower tensions, as seen above. Listed integers of say 4, 5, 7 might have been acceptable. There is no intention here to massage the data, but only to understand the implications of practical methods. Accuracy to ½mm was difficult for the 50cm arm, but 1mm would be possible with 100cm. Divisions of 2mm would be close to an ancient ligne of 2.256mm, but this unit is not essential. If the ancient strings were a longer 150cm, the arm length could be about 70cm. A design of this type could reproduce all the data seen in the lists, and without using the ‘grain’.

For the higher integers in each list, the small variation in a pair is similar to the accuracy of about 1mm with the hanger. Modern practice likes at least three readings for a good average, but the outer two are sufficient to indicate 1mm uncertainty for a single string. For the smaller pairs of integers, the inserted ½ indicates a desire for a little more accuracy, with some ability to estimate ½ a unit, or a slight bias in gauge, as suggested above. With the moderate group of integers 18, 19, 20 the inserted 19 is similar.

For basses 6 to 11 the close pairs in each list can be seen as following the same purpose, but economically omitting an unnecessary number. For example, 30,32 implies a central weighing of 31 with an error or uncertainty of 1. The average for technical performance is also 31, but there is no need to bother writing this integer. This simplification is both obvious and subtle, while our modern abbreviation would be the central figure and a plus or minus 1 (for which I cannot find the symbol). The same occurs with 38,40, then 58,60, and 73,75. Exactly the same situation arose with my device, and writing three very close numbers seemed unnecessarily laborious. In the lists, four pairs are adjacent integers and were probably better defined by the scale or weights. Four other entries are single integers, perhaps considered adequate for the ‘rough’ list. The pairs of integers may have suggested weighings of each string in a double course, but this is ruled out by pairs for the single basses and octaves, and also by the consistent closeness of the pairs.

In the initial assessment of the integers it was seen in both lists that there was precisely no overlap in the integers for neighbouring pitches, with only a single sharing of 12 between f and e. This striking point could have been repeated many times above, but can now be addressed in the light of methods of measurement. It might be a simple result of weighing errors being small enough in relation to differences between the pitches and masses. A helpful aspect is that the analysis did not need to consider overlapping distributions and averages in order to distinguish between courses. However, there appears to have been an ancient desire to ensure complete absence of overlap, and avoid confusion and contradictory specifications. One wonders whether an ancient worker anticipated or slightly massaged some numbers to achieve this.

The device was set up to find whether single short thin strings could be weighed, and then investigate accuracies. If such devices were not used, an ancient observer could still get the point, but be puzzled by the mm divisions and the hanger, then interested in the adhesive tape and unusual strings. Another conclusion is that no particular units of mass or length are necessary. For local practical use, a device only needs calibrating to suitable samples of string, and then used to select further strings. For weighing with other scales, it would have been perverse to use weights different from familiar grains, however obscure they may be centuries later, and even the next week in a neighbouring town. The corresponding description of the device is the mass distribution of the bar and the exact pivot point, so that finding an ancient diagram would be interesting but no quantitative help.

Further conclusions on weighing and string selection

The above details: on patterns in the integers, the units of grain and scruple, patterns typical of repeated weighings on an item, and accuracies of weighing, indicate a coherent method possible for ancient use. This strengthens a specification by string mass. The use of standard scales with weights seems likely, especially if one considers the integers indicate use of the scruple among the grains. However, a simple specialized device, as described above, would not require absolute units, and would have many advantages, such as easy loading of several strings, and a single side measure without the inconvenience of different combinations of weights. It is also worth stating that both lists used the same method, perhaps with the neat fuller list as an initial desired specification.

During the analysis the integers may have resembled an almost modern specification, and maybe labelling. However, the implied number of readily available strings would be at least about 80, and the implied alternatives such as 24,25 or 73,75 seem unnecessarily close. This indicates that the lists do in fact represent real weighings of strings, initially chosen by experience rather than theory, as explained above. It would then be possible to select and fit pre-weighed strings and possibly labelled items, reasonably close if not identical to the list values. As suggested above, on the simple basis of so many precisely identical integers in the two lists, it is possible that the main differences were based on new weighings, and smaller ones on estimates or labellings. (All these possible changes, and even one or both of our lists, may themselves have come from earlier lists, but this does not affect the essential matter.)

Any ancient labelling might resemble the modern situation with rare 'unrectified' gut strings. These have an almost continuous range of mass or diameter, and a maker characterizes each string by its average measured diameter. In contrast, most modern gut strings are generally put through a grinder for perfect cylindrical uniformity, and supplied in a limited number of precise gauges, like synthetic strings. This differs slightly from a more continuous range of integers for the ancient 'unrectified' strings. For selections of ancient strings, personal weighings could be suitable for larger collections of players, but labelled strings could be easier for single players and small groups.

Although this seems a plausible original use, such lists based on mass and numbers could have developed over time into a more nominal type of specification for wider distribution. Strings labelled with a required integer could be selected without any explicit reference to the original mass. This would be similar to modern usage where a player can use recommended gauges or enter tensions etc into an online Taylor calculator, without troubling with units or arithmetic. Specifications initially based on masses could later become malleable, and differ from their original meaning.

It was seen above that our lists could be an initial plan for stringing taken from a list based on pitches. This could explain why the upper octave strings 'a', g, f would be too heavy, rather than unusual choices of mass or faulty measurements. This may have been adequate and just an acceptable difference, or confusion. However, it was explained above how equal tensions with their basses, as for the lowest courses, could have been obtained with the lighter octaves such as g for F etc, perhaps indicated by the strange 'F' symbol. Other opportunities for refinements might have been the heavy upper courses 1 to 3 and the light basses A and G, but there is no written evidence. The situation here could be similar to adjustment of written schemes for fretting, whereas a set of newly tested and weighed strings would be more like the frets on surviving citterns.

An ancient string list intended to describe strings of a given sounding length for required pitches needs no other specification than mass per length. In strict technical terms our lists give no explicit information on the string material. The most probable material is whole gut with increasing degrees of twist for the lower strings. On the question of added higher density materials such as metal wire, particles or salts, the present absence of evidence is not a logical evidence of absence. However, it is likely the lists were intended to specify strings for use, rather than make a nice theoretical point. On balance, if loaded strings were an available option one might expect to see some reference and distinction. (The logic here is similar to formulating Homeric questions, or assessing Richard III apart from his 'lascivious lute'.)

One type of list that could indicate different string constructions would be integers that represent a non-technical property such as price. The price of plain gut strings would most likely have increased steadily with mass. A pronounced jump from lower integers to a higher level might imply costly, possibly specialized, bass strings. This effect would be similar to strings of different length discussed above.

<u>Course 1 string stretching</u>

For a thin top string the degree of stretching from its original length to a working tension of about 40N, is notably large and typically 50mm. However, this is small in comparison with a sounding length of 700mm, and one would not expect any large corrections. It is useful to analyse this in some detail in view of the high tensions predicted from the ancient integers. Also, there has been some recent non-scientific reference to the effect, so a decisive treatment could be helpful.

If a string has an initial cross-section area A, which is the earlier $\frac{1}{4}\pi d^2$, then when an initial length L is stretched by length x, the reduced area A_r is given by $A_r(L+x) = AL$, so that for relatively small x, we have $A_r = A(1-x/L)$. The strain x/L is related to the stress $\sigma = T_r/A_r$ where T_r is the tension in the stretched string, by $\sigma = (x/L)E$. Here E is the longitudinal modulus of elasticity known as Young's modulus, defined as the ratio of stress to strain. Now we need to find the tension T_r for which the initial uncorrected T and A produced a required pitch f from the Taylor equation. The earlier relation $f = (\sqrt{(\sigma/\rho)}\ ) / 2l$ shows that we just require an equal stress so that $T_r/A_r = T/A$ and $T_r = T(1-x/L)$. We also have $x/L = \sigma/E = T/(AE)$. The stretched tuned string therefore has a tension and area reduced in the same proportion, and given by

$$T_r = T(1 - T/(AE)) \quad \text{and} \quad A_r = A(1 - T/(AE))$$

These expressions can be made more convenient by noting that $\sigma = 4\rho f^{\,2}l^2$ to give

$$T_r = T(1 - 4\rho f^{\,2}l^2/E) \quad \text{and} \quad A_r = A(1 - 4\rho f^{\,2}l^2/E)$$

Previously one may have been tempted to make a specific numerical calculation for a slightly corrected diameter and then feed it back into a Taylor equation. This new treatment is more general and may provide some understanding of stresses, strains and moduli that will be needed below. (Refs 5,6 have other examples of general treatments.)

A calculation for a typical top string could have values of 312Hz, 70cm, 1.3gm/cc, and 3.2GPa for f ′, *l*, ρ, E. This gives the following examples for pairs of uncorrected tension & diameter:
40N & 0.45mm, 45 & 0.48, 50 & 0.50, 60 & 0.55, all with equal stress. The calculated correction factor is (1-0.078) = 0.922, and the same for all tensions since it depends on f *l*, but not on T independently of d. The tensions are reduced by a modest 8%, and diameters by 4% to 36.9N & 0.43mm, 41.5 & 0.46, 46.1 & 0.48, 55.3 & 0.53.

An important point is that the correction factor can never become large in practice because this would require large values of f *l*, or small E. The tensions would become impractically large, or diameters small, long before the correction factor became significant. For some modern synthetic materials the modulus is lower, and the large stretch produced on top strings can make their use inconvenient, so that alternatives are often preferable.

As a further example, for a top g′ string on a six course lute with 60cm and 369Hz for *l* and f at A415, the correction factor is 0.080. For A440 a semitone higher the factor would be 0.090.
For much higher pitch standards, A466 a tone higher gives 0.10.
A493 a minor third higher gives 0.115, A524 a major third higher, still a low 0.13. Meanwhile the stress, or tension at a fixed diameter, has increased by a factor of 1.56 (from $(5/4)^2$) to a level difficult to play, unkind to the lute, and perhaps beyond the breaking point for gut.

Finally for thin top strings, we can calculate the stretch x:
f ′ at 312Hz and *l* at 70cm, x = 55mm, from 0.078 x 700.
g′ for A415 and 60cm, x = 48mm; g′ for A440 x = 54mm.
g′ for A466, x = 69mm; g′ for A466 x = 78mm.

In practice these would be progressively smaller because the modulus would increase with strain, ie successive equal increments of strain require greater increments of stress, or 'it gets harder to stretch'.

The effects can also be calculated for the next highest d′ string. All we need to change here is the pitch in the correction term $4\rho f^2 l^2/E$. For the baroque lute d′ is a minor third lower than f ′ so the term 0.078 is multiplied by $(5/6)^2$ or 0.69, to give 0.054, and the corrected tensions and areas are 0.946 tines the nominal values. These are only 5% lower, compared with 8% for the top string. For a d′ string relative to g′ for A415 the 0.080 is reduced more, by $(3/4)^2$ for a fourth, to 0.045. The corrections for lower strings become progressively more negligible, but as seen below this would be offset slightly by some necessary increase of string elasticity, or decrease in E for lower strings. In fine detail, this effect would make the small reduction of tension on upper strings seem relatively smaller. This treatment is rigorous and conclusive, in contrast with the lack of firm ancient string dimensions in the main analysis. These results and the method of analysis will be helpful in following sections.

<u>Comparison of ancient and modern stringings</u>
A convenient start will be some modern schedules that are freely available on the web. These initial examples from Taylor calculators are recommended values that can be changed slightly to a player's preference.
'Gamut Strings' A392, string length 70cm, Gut:

	f ′	d ′	a	f	d	A	G	F	E	D	C
Tension N	40	38	34	30	28	27	26	26	26	26	26
Diam 0.01mm	46	52	66	78	90	118	130	146	154	174	194
Octave N						27	26	26	26	26	26
0.01mm						60	66	74	78	86	98

For A415 the same tensions would be used with slightly lower diameters:

Diam 0.01mm	42	50	62	74	86	112	122	138	146	164	184
0.01mm						56	62	68	74	82	96

‘Damian Dlugolecki’ A392, 70cm?, Gut:

	f′	d′	a	f	d	A	G	F	E	D	C
Tension N	38	32	29	25	26	23	23	24	24	22	19
Diam 0.01mm	40	47	62	70	85	113	128	143	156	166	172
Octave N						23	23	24	24	22	19
0.01mm	(46	52	66	76	92)	59	62	72	77	84	89

The brackets contain the higher diameter in a range, e g 0.40 to 0.46mm for course 1.

‘Wadsworth Lutes’ A415, 70cm, Nylgut and Kurschner (K):

Tension N	32	30	28	28	28	28	28	28	28	28	28
Diam 0.01mm	40	44	58	74	84K	112K	126K	140K	150K	170K	190K
Octave N						25	25	25	25	25	25
0.01mm						54	62	70	74	80	88K

In making comparisons with the analysis of our ancient strings, the variations of tension will be the main interest. The analysis was in terms of relative mass per length and relative tension, which are in the same ratio for a given pitch. The above diameters were included for practical completeness, but as straightforward results of Taylor calculation they are not crucial.

The useful representation of ancient tensions was a lower constant value for C to g, then a tension rising to 1.92 times greater on course 1. This allowed the examples:

18 & 35N, 20 & 38, 22 & 42, 24 & 46, 26 & 50, 28 & 54N
Lowering the course 1 integer from 5 to 4.5, from considerations of scatter, reduced the factor to 1.75 with more practical options:

20 & 35N, 22 & 38, 24 & 42, 26 & 45, 28 & 49N

Further reduction to 4, as indicated by methods of measurement, gave a factor of 1.54, which would widen the practical possibilities to:

20 & 31N, 22 & 34, 24 & 37, 26 & 40, 28 & 43, 30 & 46N

This could lead to many long lists of possible tensions and diameters or masses, but the lack of information on absolute mass per unit length, pitch standard and string length cannot lead to a single certain scheme. A useful approach is to base examples on the modern schemes, mainly to compare variations of relative tension and hence mass and diameter over all 11 courses, but without insisting on identical absolute values. It is worth noting that all these modern stringings have been found suitable or optimum for certain lutes and strings, and many other makers and players use similar recipes. Similar variations of tension are also suggested for the six courses of vieil ton, usually with lower tensions for the double course 2. Here, uncertainties with high pitch standards may imply much higher tensions beyond modern comfort, especially for an ancient double top course. A difficult problem may have been bypassed by adopting a limit of the modern A440, rather than several semitones higher.

Starting with the Gamut A392 values, the five basses G to C are all at a constant tension of 26N, and notably the octaves are also at this tension. The tension increases slightly up to 30N for f on course 4, with an equal octave on A. Tension then rises steeply for ‘a’, and up to 40N on the top course. This is almost the same as our possible 26 & 40N. The ancient tension increased less between G and f, but the slight increase developing before courses 1 to 3 was particularly evident in the higher tensions on ‘a’ and g as octave strings. This single example shows that all possible cases for a course 1 integer modified from 5 to 4 have about same relative variations of tension across all the courses as the modern case. It also shows that with an unmodified top integer of 5 there is still a constant tension for lower courses, but that the final rise to 40N would only be compatible with a lower constant of about 21N. An integer relaxed to 4.5 would require a lower constant of about 23N. Many modern players would consider this too slack, but see next. (A reader may already be inquiring about the calculations of tension reduction on the upper courses. So far all our tensions are uncorrected, so the comparisons are valid. Also the corrections were shown to be small, and their main use will appear below.)

The ‘Dlugolecki’ values are even closer to some ancient options. Now the six lowest basses are effectively at an approximately constant tension, with octaves equal to basses. A small change such as 23N for E would require a small decrease from 1.56 to 1.49mm. The slight variations in tension may reflect availability of gauges, rather than a strict recommendation. The dip to 23N for A and G looks similar but smaller than the ancient case, where available masses were also a possible cause. The final 19N for C appears to be set by the maker’s limit of 1.7mm diameter for fitting strings on a peg, a practical effect appreciated by anyone who has used bulky gut basses. The tension rises for d on course 5 but then dips slightly for f before rising sharply to the top string. These detailed alternations for the 8 lower courses might be characterized as a slight gradual increase from 19 to 26N, or even an approximately constant average of 23N. The increase from course 4 to the top 38N is similar to the ancient integers, with a tension ratio of 1.63, slightly less than the option of 22& 38N for a top integer modified to 4.5. (One can calculate that the diameter for 38N is 0.45mm, near the upper quoted range.) This modern schedule is therefore closer to the ancient case than ‘Gamut’ on account of its lower bass tensions, and maybe the slight dips. All the octaves have the same tension as their basses, like Gamut, but in contrast with some unmodified ancient integers.

The final comparison is with the ‘Wadsworth’ values for modern materials, but probably also intended for gut. The higher pitch can be compared with the Gamut alternative. The schedule has many differences from all three cases examined above. The lower nine courses are all at 28N, but the six octaves all have a lower 25N, and courses 2 and 1 have just a small rise to 30 and 32N. Stringings of this type were adopted in the initial modern lute revival, and often with a perfectly constant tension of about 30N across all courses. In the example the tension increases but with a very small factor of only 1.14, very different from the ancient case.

Over recent decades an initial constant tension stringing with modern materials gradually led to the type seen in our first two examples. The basses generally had wire windings, adopted from guitar strings, and a strong sustained sound. This could be quelled by lowering their tension, to about 26N, with a lower gauge. Conversely, upper strings were relatively ‘underpowered’ at 30N and could be improved with tensions of 35 or 40N. There have been several subsequent strange ‘twists of history’. Firstly, the constant tensions with synthetic strings were modified as just described. Secondly, the modern improved varying tensions with synthetic materials were found useful for modern gut. Thirdly, these modern tension variations might not have been so necessary for gut, since lower strings may not have needed so much quelling nor upper strings boosting. Fourthly, the modern tension variations now appear similar to an ancient use. Fifthly, the ancients may have used even greater ranges of tension. These complications have become tangled with notions from ancient writings of equal (constant or even) tension or feel. Some aspects need to be addressed below, but a later full appreciation will take many more pages.

The ancient integers are compatible with a wide range of stringings, but there is not enough information to be more precise. A key aspect of any set of strings is their absolute level of tension and hence mass. Even a small change, equivalent to a semitone or a factor of 1.06 in diameter, or 1.12 in mass or tension, can make a great difference to a player and also alter the response of the lute itself (eg Ref 10). Knowledge of absolute ancient levels would be of equal, or even greater, importance as the relative variations across the strings, but accuracy to 10 or even 20% cannot be produced without resorting to insecure assumptions. This may not be too dreadful, because apparently full details could have led to a single quick solution that became widely accepted, but was actually unrepresentative, incorrect or impractical. The lack of data has forced the present study into many lines of inquiry, leading to a reasonable range of solutions, and with the bonus of several related conclusions on the way.

A single full example of ancient possible stringings may be sufficient, and easily adapted to cover the entire range of possibilities. It is useful to take some points of reference similar to the modern schemes. Then we can at least be assured that the examples would feel practical, and players could easily try them. Ancient reality may have been more extreme, but this is inaccessible to us.

Firstly, a lower constant tension of 24N will be used for C to f, which conveniently includes our earlier F string example, and the moderate practical level unexpectedly found later in 'Dlugolecki'. This 24N leads to a rapid rise from 'a' to f ′ at 46N, by a factor of 1.92 using the top integer 5:

'Coakley from integers' A392, string length 70cm, Gut:

	f ′	d ′	a	f	d	A	G	F	E	D	C
Tension N	46	38	29	24	24	24	24	24	24	24	24
Diam 0.01mm	48	53	61	70	84	112	124	140	149	168	186
Octave N						29	25½	24	24	24	24
0.01mm						61	64	70	74½	84	93

The octave strings 'a' and g are taken from the original integers. Some care is needed to calculate g slightly above the tension rise. However, 24N with 0.56 and 0.62mm may be better for these two octaves, and perhaps also the real ancient preference. As discussed above, they might have used the alternative integers 9, 8, 6 for the octaves on F, G, A to give tensions close to the basses.

Alternative tension rises to 42 and 37N, corresponding to top numbers 4½, 4, suggested above, are also of interest. For the d ′ course, reductions from 6 to 5.66 and 5.33 retain the initial proportions relative to a fixed 8 for 'a', avoiding unwarranted and messy changes to 'a', g, f:

	f ′	d ′	f ′	d ′
Tension N	42	36	37	34
Diam 0.01mm	46	51	43	50

This single example can be extended to any other tension levels permitted by scatter of the integers, within and between the two original lists, and by uncertainty in the four scalings for absolute mass or diameter, pitch standards, sounding length and density. The 24N lower constant could be changed to any other practical level between about 20 and 28N. For example, 20N scales the above higher tensions and masses by 0.833 (20/24), and the diameters by 0.913 (from √0.833). The large relative rise for higher courses severely limits possible values for the lower constant tension. Many of the higher courses will be considered too taut and many lower courses too slack, with a relatively narrow suitable central band. The quite low 24N dictates an unmodifed top 46N, with possible reductions to 42 and 37N, but a lower 22N could reduce the 46 to a more reasonable 42N. Tensions that initially seem too high or low for modern use should not be dismissed too hastily in paper exercises.

Several detailed effects could help to reduce problems with the large factor of tension from basses to top string. Plain gut basses are bulky so that lighter options are preferable if not too slack, which is also a consideration in the modern schedules. For the top string, all diameters break at the same stress and hence pitch - according to simple theory expressed in the initial equations. However, it seems likely that thicker strings would be less prone to small defects and may have slightly higher breaking stresses. Also, in the 'rough' list of integers, the basses are heavier by a factor of about 10% (eg 73 to 82). Their tension would be 10% higher than the lighter neat list. A high top string tension is determined by its integer so that the heavier basses would reduce the large tension rise factor, by 10%. (A quick glance at the larger ratio 82/5 might initially suggest that the lighter option with 73/5 would be better.) An optimum balance would be difficult to calculate, so fortunately the effects do not seem to be strong, but useful differences might be found with real strings. These matters have some relation to whether one list was the basis for changes in the other.

Interested players who may have problems using the Taylor equation could try the calculators on the web. These require inputs of tension, string length, and pitch standard, and then output the string diameter. Usually the range of allowed inputs is limited, and the density fixed. A useful simple modification of the calculators would be an output of any one of the five quantities in the Taylor equation when the other four are inputs. At present, if one wants to find a tension needed for a given diameter, as for the ancient integers, then a series of trial tensions could be used to 'home in' on the known diameter. Many other neat tricks are possible, similar to my use of a basic C40 tuner for any desired temperament in Ref 5, but this may need more science than using the Taylor equation.

<u>Useful and available string gauges</u>

An interesting aspect of both the ancient and modern strings is the availability of different gauges, and particularly the increments of mass or diameter. The ancient integers ran from 5 to 82 with an increment of 1 unit. This implies at least 78 specifiable, and perhaps available, masses per length. An increment of mass per length $\Delta\mu$ indicates a highest relative value $\Delta\mu/\mu$ of 1/5 or 0.2, which is the lowest resolution or accuracy of 20%. The lowest $\Delta\mu/\mu$ is 1/82 or 0.012, and the highest accuracy of 1.2%. The important consequences of the very coarse or inaccurate definition of thin strings, and conversely the excessively fine definition of the thick strings were discussed in analysing the ancient lists.

Relative increments of diameter $\Delta d/d$ are ½ of the $\Delta\mu/\mu$ values, and can be compared with the available values for some modern strings:

	$\Delta d/d$ treble	$\Delta d/d$ bass
Integers	1/10	1/164
Pyramid	1/16	1/60
Aquila	1/20	1/36
Gamut	1/20	1/90

The Pyramid values come from 0.025mm increments near diameters of 0.40mm, and roughly 0.03 steps near 1.8mm equivalent diameters, for which there are also several types of construction. Both ends of the range are better, with finer treble and coarser bass increments, than with the ancient integers. The Aquila values come from 0.020 steps near 0.40, and 0.05 steps near a 1.8 equivalent, and both treble and bass are even better. The intermediate strings have graded increments of 0.03 and 0.04. The Gamut increments seem to be 0.02 over the entire range, with many other types of construction. This bass accuracy approaches the ancient case.

The relevance of these comparisons is that one might consider an ideal resolution would be constant across the whole range. This is because the relative resolution of pitch $\Delta f/f$ is equal to $\Delta d/d$ and two criteria seem useful: an adequate resolution in the treble and the same sufficient resolution down all the lower courses. In practice the 0.025 steps for Pyramid can seem too large, as in 0.425 or 0.475 alternatives for 0.45mm. On this basis Aquila and Gamut trebles are better but the bass steps could be increased, perhaps to the same 1/20, which is 0.9 or 1mm. This would seem about adequate for basses in all the above schedules, and Aquila is closest with double the minimum. For gut strings the unexpected extra gauges may make use of the range of natural variations of size, whether ancient or modern.

Another aspect is the number of strings needed in a range. This can be counted, or estimated from the steps. The ancient range is around 80, but with a method of manufacture and supply different from modern items, this may not be completely available nor necessary. For Pyramid, an average of say 4½ per 0.1mm range from 0.4 to 1.8 gives 59 items, and with different types I counted 78. For Aquila, about 3½ per 0.1mm gives 47items, and I counted 49, plus 10 wound types with the same gauge as monofilaments. For Gamut the estimate is 66, which does not need counting, and several other types of range. The number of items for the uniform optimum resolution is only about 40 items, which is not a great saving on the Aquila range.

An analysis of ancient integers using a precise calculation with no smoothing of scatter produces uneven variations of tension and diameter (eg Ref 1). However, with smooth variations of tension, as used in my analysis or modern recommendations, the calculated smooth variation of diameters may also be difficult to match to the available stepped diameters. The effect can be quite random, sometimes being almost perfect, but often requiring awkward choices even when reasonable increments are available. Another additional effect here is rounding of the calculated diameters usually seen in written schedules, and above. A final stringing could be rather different from the intention, even with modern methods and strings. (There are some apparent discrepancies between the four schedules above, so we should not be too disappointed if 300year old sparse evidence seems a little uncertain as well as obscure.)

<u>Feel of strings</u>

At this point we need to raise the question of ‘feel’ and its relation to tension. Firstly, we can note that for an 11 course baroque lute, courses 1 and 2 usually had a single string, and courses 3 to 5 had two strings, presumably at the same tension. Courses 6 to 11 were also double strung but with a bass and an octave, sometimes with slightly different tensions, as seen in the ancient integers or less taut modern octaves. Many ancient writings recommend an equal or even feel for the fingers across the courses. A modern technically minded reader might think this is some ancient specification for all strings at the same tension. Then one realizes that ‘feel’ would be affected by many other factors. However, this could lead to a modern delusion that a simply expressed ancient qualitative desire for a comfortable feel involves some complicated ancient algorithm left for solution by modern players.

Seven years ago I studied the lute mechanics affecting ‘feel’ (Ref 10). After considering the importance of physiological, psychological, and historical aspects, I decided not to wander directly into these even more uncertain areas. Two features of lutes illustrated the difficulty. In the Gamut scheme, the single top string has a nominal tension of 40N, and course 2 also has a single string at a slightly lower 38N, whereas course 3 has two strings each at 34N. This produces a large discontinuity in total load on the fingers and arm, 38 to 68N. In contrast, for vieil ton course 2 is double, but often at a lower 30N, producing a differently positioned discontinuity of 40 to 60N. Players grow accustomed to these large differences, and to changing lute types. A contrasting feature of lutes is that even small differences of tension and string type between any courses can feel distractingly uncomfortable.

These observations raise the central question of whether an equal or smooth feel involves the total for a double course, or just one string, in relation to a single course. At one extreme, for an equal contribution from each string, then the ‘initial modern’ or ‘Wadsworth’ type is close to an equal tension with 32, 30, 28 plus 25N octaves. At the other extreme, for an equal contribution for each course, an ancient type such as ‘Coakley from integers’ is close with 46, 38½, 29, 24 plus 24N octaves. An exact limit would be say 46 for courses 1 and 2, with 23 for all other strings. As seen above, neither case appears to be optimum, the ‘initial modern’ tending to slack light upper strings, and the strict ‘ancient integer’ tending towards heavy taut top strings or slack lower strings. ‘Gamut’, and especially ‘Dlugolecki’, steer a middle path, with reasonable closeness to the ancient integers and clearly a generally comfortable feel. Many other factors will be involved such as: reduction of tension due to stretching, pressure and area of contact on fingers, bending stiffness, a plucking or stroking action, a sensation of size or bulk of string, anticipation of sound quality, and historical sense (the ancients never heard nylon or our loud metal windings). These factors are significant, but probably secondary to the tension and number of strings, and will be presented quantitatively in a future paper. Experiments would be important but probably little use without a theoretical framework. In the present context, players may like to compare smooth variations of tension across strings with calculations from the scattered integers or Ref 1.

However, a recent view of feel, on the Aquilla website, quoted by other suppliers, and mentioned in the summary of Ref 1, seems unhelpful. The main purpose appears to be explaining away the increased upper tensions found to be useful in many modern schemes, and now in the ancient integers, because this feature conflicts with a simple modern view of an ancient constant tension or feel across all courses. Most secondary factors tend to increase the sensation of tautness in thin strings, so the websites contend that the thinning effect would reduce the tensions. As analysed rigorously above, this is qualitatively true but the effect is at most a small 8%, and in attempting to reduce a nominal 40N to perhaps 24, a mere 3N is insignificant, and the venture is unhelpful. With all the mechanisms listed above it would be essential to make analyses that can compare the relative size of the effects, and most usefully the dependence on other physical quantities. Equal feel or tension is a far simpler concept on a violin with four single strings, and the above calculation of stretching produces a small correction factor almost identical with the lute value. For a lute, however, the notion cannot be expected to apply simply, or at all, to single courses next to double ones.

Stretch sharpening and angled bridges

A further interesting aspect of ancient strings, also mentioned in the Ref 1 summary, can be outlined here while we have in place all the necessary physical relations, such as string stretching and pitch dependency on stress. This will also demonstrate a powerful use of physics, in contrast with the string schedule and its sad deficiency of data in an otherwise simple situation.

It is well known that some bass strings can have insufficient elasticity such that fretted notes are sharpened uncomfortably when a string is stretched slightly on the frets. The effect depends on the amount of stretching on a fret relative to the initial stretch required to tune a string up to pitch. The sharpening can be corrected roughly by slanting the frets towards the nut, which is not seen in ancient paintings or writing, and disliked by modern makers. Another method is to angle the bridge to produce slightly longer bass strings, as used on some modern guitars.

On many surviving baroque lutes, bridges are angled such that the basses are shorter than the trebles. About thirty years ago, David van Edwards valuably highlighted this feature and summarized the surviving variety of shorter, longer and equal length basses (Ref 11). The notion also arose that the lutes with shorter basses would require a very inextensible top string. This was thought to cause stretch sharpening much greater than normally found on rather stiff bass strings, and by analogy with longer stiff basses was suggested to permit tuning of shorter basses. This poses a problem for all the intermediate courses, even before examining the physical plausibility. Some practical lute makers and players seem to have accepted the notion, without any analysis or practical demonstration: but it does have a technical feel, and elasticity is often at the root of string problems. Eight years ago I set up the general problem of a graded increase of stiffness from bass to treble strings (Ref 6). It became apparent that prohibitively large treble and low bass stiffnesses would be required.

Instead of relying on specific cases I therefore sought a more fundamental analysis of this practical obstacle. This was given in Refs 6,9 but perhaps the colloquial terms, instead of helping readers with limited science, appeared as just another vague opinion for debate or controversy. The best way to present the result is a short rigorous technical proof, which should be helpful to those with some physics who may be new to the problem, or have been misled by the initial notion. The physics is simple but the main effort was realizing how to analyse the problem. (Philosophers have invented the term ‘bridging concept’ for something as natural as breathing for practising scientists and student exercises. The ‘laws of physics’ usually tell us nothing directly about interesting practical situations.) The key to the problem is finding the elasticity of basses and trebles that would be needed for the shorter basses to have very little stretch sharpening and hence large initial stretch, relative to much greater sharpening and low stretch in the trebles. This is the opposite behaviour of all strings presently known to mankind.

The solution takes a treble string and a bass string 2 octaves lower, for example f ′ and F on a baroque lute, or g′ and G on a 6-course lute. An intermediate f, or imaginary g, may also be helpful. Lower baroque basses are unfretted, but a fretted D below G increases the elastic problem. A ‘5 line proof ’ can be given in a short table describing the mechanics of strings. Initially, in lines 1 to 3, all strings are of the same material and same modulus of elasticity. The modulus is the material and constructional property (eg twist) of the strings we are interested in examining. Explicit tensions and diameters are unnecessary, but could be added for illustration.

String	f ′	f	F
Line 1. Relative pitch	4	2	1
2. Relative stress	16	4	1
3. Relative stretch	16	4	1
4. Elastic modulus for equal stretch in f ′ and F.	16	4	1
5. Elastic modulus for stretch of F 16 times f ′, ie reversed stretch.	256	16	1

Highly technical readers will immediately see in line 4 the impractical range of elasticity needed to make trebles fret-sharpen by the same degree as the basses. They will also see in line 5 that for the initial degrees of fret-sharpening to be reversed, so that trebles sharpen like the initial stiff basses, and the basses like the initial trebles, then the requirement for basses to be 256 times more elastic than the trebles is prohibitively large and practically ‘completely impossible’. A necessary grading of elasticity for all the intermediate strings compounds the difficulty. (A further difficulty arises because the low modulus for the highly stretched bass is at a high position on the stress/strain curve, so the required low strain value is even lower. A small factor offsetting the difficulty is that bass strings have a slightly higher action height and hence greater stretching by a fret.)

For less technical readers the following explanation may help:
Line 1 gives the ratios of pitches an octave apart, and should be clear.
Line 2 follows from $f = (\sqrt{(\sigma/\rho)}) / 2l$, giving the stress an octave lower as ¼. Here all the relative values have been based on a lower 1, for simplicity. A line of 1/16, 1/4, 1 is less clear.
Line 3 gives the initial stretches of string when winding up to pitch. These follow from the relation between stress and strain seen earlier. The modulus of elasticity is stress divided by strain, so strain is stress divided by modulus, or $x/L = \sigma/E$. In this line all the strings have the same modulus and initial length, so the stretches are simply proportional to the stresses.
Line 4 shows how one needs to change the modulus so that all the strings have the same stretches. This is just proportional to the stress, from the above relation, so that the bass F needs to have a modulus 1/16 that of the f ′ string.
Line 5 then shows how one needs to further increase the relative modulus of the f ′ string so it has a stretch 1/16 that of the bass F. The required modulus of the bass F string is 1/256 that of the treble f ′ string. This reverses the initial relative stretches of the treble and bass strings.

The familiar initial position in line 3 of relatively large treble stretches requires large deceases (1/16) of the bass modulus to produce identical stretches in the bass in line 4. Even this stage would be practically very difficult. In order to reach the state in line 5, where the bass string has a much larger stretch than the treble, requires yet another 1/16 lowering of the bass modulus. This would be an ‘extremely unusual’ practical situation, requiring a bass modulus 256 times smaller than the treble value. In practice one would need a bass material similar to a very soft rubber, typical modulus about 0.03GPa, and with a high strength, together with a treble like a hardwood near 10GPa. Shifting the absolute values, for a soft plastic bass at about 0.3GPa, the treble modulus would need to be around 100GPa for steel, but with a low density. These are the nearest classes of common materials one can find in data tables. Both these examples, and the grading of intermediate strings are effectively impossible for ancient materials, however ingeniously gut was processed, and maybe even with today’s materials for the rubber strength, and low density/high modulus.

Readers with little science may find a simplified practical account useful. For typical modern strings the initial treble stretch is around 40mm. Even for basses overwound with metal wire the stretch is a rather limited 10mm, still with a small trace of stretch-sharpening, and modern gut basses can be worse. One can see that increasing the stretch to the 40mm treble value would be ‘extremely difficult’. In addition to material properties, the extra details of construction such as twisted metal wire do not allow the same degree of reversible stretching possible with a simple homogeneous treble string. The next stage of increasing the relative bass stretch to 160mm, or equivalently 40mm relative to a treble 10mm, should be inconceivable to practical makers of lutes and strings.

Today’s string makers suggest various stringings, from Taylor calculations and testing, and modern lute makers and players can use these or make small changes. They can also compare quite subtle aural differences and express their preferences, often quite strongly. However, these musically useful strings reflect a very narrow range of physical behaviour. Just as expert wine making does not require, or confer, deep general knowledge of organic chemistry, so trying out modern strings does not provide guidance outside this narrow range of elastic behaviour. The ‘elastic notion of angled bridges’ turns out to be analogous to a pilot of a propeller driven aeroplane, accustomed to low altitudes, then conceiving the notion of flying to the moon and maybe still believing there will be sufficient atmosphere; or an untested suggestion for a country walk where one direction ends down a pothole and the other meets a precipice.

In view of the impossibility of the elastic notion, I realized the only other possibility was some geometric variations in strings. In Refs 6,9 the simplest possible variation was seen to be a taper down the length of strings, and the acoustic effects were examined in some detail. It is not appropriate to repeat this here, but I have since done some experiments. These fully support the chief predictions: (i) that tapered bass strings which are thicker towards the bridge can remove stretch sharpening on the frets, and (ii) increasing this taper allows one to shorten the string, using a simple false bridge in the experiments, and retain correct tuning on the initial frets. The long-range tapers involved are small, of order 2%, which is just 1.50 to 1.53mm, between nut and bridge. Turning the string around destroys the tuning, and written sources advise reversing strings to improve tuning. This is effectively a ‘taper detector’.

Ancient gut strings were produced in large-scale handmade processes. Natural guts were, and still are, slightly naturally tapered, similar to trees and many other biological structures. This long-range taper would tend to persist in the finished strings, however carefully the natural guts were processed. In the modern lute revival, players started with synthetic strings of uniform diameter, or mass per length, and they began to expect uniform gut strings. Most modern gut strings use strips and any non-uniformity, including residual taper, is removed by grinding between wheels (see descriptions on makers’ websites). There may be some specialists who try to make a uniform lute string without this ‘rectification’ but this would be a time-consuming and potentially wasteful process, and very costly or uneconomic in contrast with the large-scale ancient manufacture of naturally tapered strings. In modern times, ‘unrectified’ strings have found wider use for bowed viols etc, rather than sensitive plucked lutes.

One of the earliest lute books, by Capirola (Ref 7), states quite simply and repeatedly that lute strings are tapered, with no hint of special curiosities. The previously cryptic written instructions for fitting strings are an exact parallel with my theory and experiments. A further result of taper is ‘false’ strings, a constant ancient preoccupation in their standard test of holding a potential string between outstretched hands. Sources indicate the ancients were content to match together slightly false double courses, which were at least as useful as one false and one perfect string. However, this test would only detect and sift out strings with gross local irregularities, but not those with slight gradual taper. I have calculated the effect of taper, and also detected a slight effect in the experiments. The inharmonicity increases with taper, indicating that ancient lute sound might not generally be the perfect, and sometimes bland, modern ideal. This effect also provides a reason for an upper limit to the bridge angle or shortening of bass strings by about 6mm, which is also found for surviving lutes. In addition, the tuning corrections, of order ¼ semitone are linear only up to the same limited degree.

The final view in Ref 9 was that tapered strings were the general ancient item, resulting from the natural gut structure and large-scale manufacture. Uniform strings would have been a rare sub-class. The ancients learned how to use these tapered strings to correct stretch sharpening, to allow good tuning of many sorts of angled bridges and necks, and to fine tune octave strings. Some misunderstandings, or my explanations, may have troubled adherents of the elastic notion. For example, the ancients would not deliberately angle bridges in order to tune tapered strings. Nor would they generally need to make special tapered strings in order to tune angled bridges. However, the phenomenon of angled bridges may have remained overlooked without the much earlier contribution of Ref 11, which triggered my interest leading to the role of taper for ancient strings and lutes.

Concluding Remarks

In finally signing off, a list of short points would be difficult, but the many section headings, and the summary guide for Lute News should be helpful. It has been possible to interpret many aspects of the ancient integers and their patterns in terms of methods available to the original workers. One striking conclusion was particularly helpful in analysing the integers. The accuracy and usefulness of the integers is lowest for the small ones on the upper strings and greatest for the lower strings, exactly the opposite of a first impression, and this transforms an awkward set of data into something more intelligible and practical. Another intriguing deduction is that the ancients appeared to be using the listed integers to produce improved specifications such as octave strings, and these are also contained within a list.

A further piece on ancient strings and barring of lutes had been planned, and may now be resumed. This can include the new conclusions from the ancient integers, but as seen above this will not be so decisive or unusual as the ramifications of taper. The question of why bridges are at an angle requires tuning in to ancient thought, as in the present work.

Footnote

Just before submission of this paper I was told that similar integers had been found in Diderot’s Encyclopedie, which appeared in about 1750-70, at least 80 years after the manuscript. This seems to throw my ‘preliminary matters’ at the first fence, but it would be surprising if the whole exercise were invalidated. At times throughout this work it was noted that the integers had a more modern feel than earlier trial and experience methods for fitting strings, and even resembled a form of labelling. However, at this later date ancient units were still apparently unstable, with no further hope of deducing mass and length units, unless some key measure comes to light. It will be interesting to know more, perhaps how our odd pages of integers became attached to the music. Continued interest in the 11 course lute and the 80?year old music of Denis Gautier would be unusual in those times.

After submitting this paper, Damian Dlugolecki's view of the integers appeared in Ref 12. There is no suggestion that the lists are later additions, but just that a system of numbering strings from 1 to 50 was mentioned by Diderot, and may or may not have been used 80 years earlier.

Appendix on Twisted Strands

The above sections on Diameter of String, Strand Number, Mass of String made distinctions that are crucial in understanding the lists of integers. My first draft contained a discussion of the effects of twisting strings in interpreting the integers. This would have seemed too lengthy, abstract, and critical of the results in Ref 1, so I summarized the problem briefly, and then proceeded with some positive analysis. The two sentence summary was 'A more devastating problem (for a description of strands) is that real strings would be twisted. A probable graded increase of twist for lower strings would hinder calculation of relative string diameters, and assumptions of twist from modern strings would not be reliable.' A deadline for Lute News did not allow a later explanation, and at the time this did not seem too necessary. However, some readers and also Ref 12 may have thought that the stringing in Ref 1 takes account of twisting, or that its effects can be neglected. Rather than a complicated verbal description that I first drafted, the matter can be presented more clearly by adapting the original lists of integers:

Course		Integer simplified MS list	Mass&T this work & ref 1	Mass equal twist	Mass&T twist x 1.2	Mass&T twist x 1.5	Mass&T twist x 2.0	Integer for T as in col 2
1	f ′	5	5, 1.92	5	5, 1.92	5, 1.92	5, 1.92	5
2	d′	6	6, 1.6	6	6.1, 1.63	6.3, 1.68	6.6, 1.76	5.5
3	a	8	8, 1.2	8	8.3, 1.25	8.8, 1.32	9.6, 1.44	6.6
4	f	12	12, 1	12	12.7, 1.06	13.8, 1.15	15.6, 1.3	9.2
5	d	16	16, 1	16	17.3, 1.08	19.2, 1.2	22.4, 1.4	11.4
6	A	24	24, 1	24	26.4, 1.10	30, 1.25	36, 1.5	16
7	G	32	32, 1	32	35.8, 1.12	41.6, 1.3	51.2, 1.6	20
8	F	40	40, 1	40	45.6, 1.14	54, 1.35	68, 1.7	23.5
9	E	48	48, 1	48	55.7, 1.16	67.2, 1.4	86.4, 1.8	26.6
10	D	62	62, 1	62	73.2, 1.18	90, 1.45	117.8, 1.9	32.6
11	C	78	78, 1	78	93.6, 1.20	117, 1.5	156, 2.0	39

The present interest is the full range of integers, but simplified values are sufficient to illustrate the important effects, and details of the octaves can be left out. The first column has the simplified integers. The second column repeats these integers as representing mass per length, and also the derived relative tensions, from the constant level on the lowest eight courses rising to the top course 1.92 times higher. This would also hold for Andreas Schlegel's calculations, which in effect assume that the strands are simply collapsed into a uniform cylinder. Some readers have thought effects of twist were included, but this is not so. (It has since emerged that a brother of Andreas made the calculations so that the technical and musical distinctions are not under the same hat.) The third column introduces twist in a completely neutral way, with all the different courses given the same amount of twist and hence retaining the same relative mass per length and tension as in the previous column.

The remaining four columns introduce twist that realistically increases from the top course towards the lower courses. The first case has a steady increase in twist from f ′ to the low C string with 1.2 times the mass without twist. A simple linear increase is used for illustration and this can be neatly arranged with 10 simple steps between 11 courses. The lower courses become steadily heavier and thicker than given by the original integers in a description of mass per length (or strands that are untwisted, or have equal twist). This is the consequence of adopting a strict description of strand number and then realistically including twist rather than neglecting it. The most significant effect occurs in the list of tensions, which are steadily increased for lower strings by exactly the same factor of mass increase resulting from the increasing twist. The sharp rise for the top course is slightly less, by a factor of 1.81 (from 1.06 to 1.92), but the main effect is the steady rise for lower courses by a factor of 1.13 (from 1.06 to 1.2). I saw there would be a problem for real strings, where the graded twist, from low to medium to high, could be considerably greater.

It is helpful to outline here how the mass per length depends on the degree of twisting. If the twisted strands make an angle θ with the central axis then the initial untwisted length l is reduced to the shorter twisted length $l_t = l\cos\theta$. The mass per length is therefore increased by a factor $1/\cos\theta$, since the volume, or cross section area times length, remains constant. The above factor of 1.2 corresponds with an angle of 33 degrees. (One frequently sees lengthy expressions for properties of twisted strings. These may look impressive, but they are ideal cases originally derived for metal wires, where neglecting deformation of strands and friction may be reasonable. Gut strings are very different since they resemble a wet cloth rung out such that the strands are deformed into complicated interlocking shapes with no air gaps.)

The next two columns use values of a 1.5 and 2 fold increase of mass, or about 1.2 and 1.4 in diameter, arising from twist angles of 48 and 60 degrees. The tension rise factor for the top course is reduced a little, but the tension rise for lower courses becomes as large as for the top string, with a strong dip towards course 4. This would be unsatisfactory for real strings, and strongly indicates that the strand notion in conjunction with realistic twist would not be a correct interpretation of the integers. My summary in Ref 12 understated the effect of twist as ‘hindering straightforward calculation’ of mass per length. Signalling the full implication by ‘devastating problem’ might have been unclear.

Another way of demonstrating the problem is that a low C with an integer of say 78, when twisted with the 1.5 factor, would behave as an untwisted string with an integer of 117. This would have a tension 1.3 times the low level on course 4 (from 1.5/1.15). For a string to behave as a mass description of 78, the integer would need to be 52 (from 78/1.5). The effect could be especially strong because the original integers for C and D are already relatively high and in the mass description lead to slightly higher tensions than the strings above, rather than a slackening often seen in modern schemes. For higher strings the difference is proportionally smaller, and an F with an original integer of 40 would need to be about 30 with twist.

A final illustration is in the last column which gives original ‘integers’ with 2.0 twist factor that would be needed to produce the constant level of tensions found with the mass per length description (or strands with no twist or equal twist for all courses). The difference becomes increasingly large up to course 11.

The assumed steady increase of twist helps illustration, but a stepped increase shows the same effect, although less clearly. Here one might inquire what amounts of twist would be needed to make the tensions from calculations without twist resemble modern schedules. Their more gradual increase in tension from course 6 to 1 would require a twist that was low for the upper courses, but high for the middle courses, and then moderate for the lower courses. This is highly unrealistic since it would not be useful for moderate increases of elasticity on middle courses and larger increases for basses.

The final strong implication is that the original integers in the lists are not compatible with a realistic representation of twisted strands, and a description of mass works much better. The important difference is that mass per length already includes the effect of twist, and the integers lead to a reasonable variation of tension across the courses. In contrast, a description of strands would lead to large increases of tension for lower courses using the listed integers, and much smaller integers would be needed to give reasonable tensions.

Some general aspects of the two descriptions, by mass or strand number, can be compared. An attraction of strands would be a direct definition of gauge, diameter or mass if only one knew the exact type of gut etc. This would, however, be concealed by the addition of twist. Twisting shortens the initial length such that the volume (Al_t) remains constant. Here the variations of both A and l_t would be unknown for us centuries later. The parallel problem in the mass description is deducing mass per length from total mass (μl). Here, although we do not know the length, the assumption of a constant value makes good sense. An ancient musical compiler of the lists could be fully aware of a single standard length, but less informed of other variations in a strand description, and this would lessen its usefulness. It may therefore be fortunate for us that the strand description appears unrealistic. In addition, many fine details of the integers were explained convincingly in the mass analysis, but this does not appear possible in a description of strands.

A strand description may appear practical and natural for a maker of strings, but less helpful to a user unaware of all the possible variations. In contrast, mass per length may seem a modern scientific concept, less likely for an ancient description. However, Mersenne used mass decades before our integers, but he was primarily an expert in abstract number theory, rather than a pre-Newtonian theoretical physicist. It is likely that mass or weight was an existing practical characterization for strings, also used in sonometer experiments, and adopted by Mersenne. Ancient string makers and users had been perfecting their craft for centuries, and were aware of many useful physical correlations, including tapered strings, unknown to the initial lute revival.

Ref 12 notes that there are no other ancient writings on strand numbers. Many of the ancient snippets are so cryptic and Shandyesque that they can be combined at will to prove that ‘black is white’. However, in Ref 9 I noted that one of Dowland’s few firm remarks is that the ends of a treble string have two strands, in contrast with the integer 5. Diderot’s 1 to 50 numbers, whether mass, strand or something else, may remain obscure rather than helpful.

At present I would leave the matter as a question or a friendly challenge, since the above examples of twist are representative rather than firm. It would, however, be interesting if a string maker could present a set of strings for an 11course lute based on a strand description. The present analysis suggests the required integers would be very different from the ancient lists, unless some unusual changes of strand type compensated for effects of twist. In particular, with a surprisingly large number of 5 thin strands on a top string, the neglect of twist would already indicate an enormous number of about 80 identical strands on the bass C. Realistic inclusion of twist would then require progressively thinner strands for lower strings, which appears even more unpractical. Finally, it can be added that all the physical analyses in this paper may collectively form something more coherent and decisive than an isolated assumption of a string property in the equations for pitch.

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