DOWLAND’S LUTE TUNING and other ancient methods, including GERLE’S

It is three years since I last wrote here, and recently further thoughts have developed. The initial interest was John Dowland’s temperament - but not for once his ‘melancholy’. This was given in terms of fret positions in his son Robert’s ‘Varietie of Lute Lessons’ of 1610 (Ref 1). The scheme has often been passed over, but without full examination and explanation. I considered a comment for the earlier Ref 2, but did not then have a neat treatment. People continue to be puzzled by the writing - recently Nigel North during his Lute Society talk in November 2010 (Ref 3), soon after recording Dowland’s complete solo lute works. At first I thought a short technical assessment of the usefulness of the tuning would be sufficient. This now forms the next three sections of this long paper, and initially it was to be combined with other work, in a brief four pages.

I began to realize that a full treatment might be possible for Dowland’s fretting instructions. (Fretting or worrying is an aspect of melancholy.) During this research an equally important result has been an explanation of how Hans Gerle derived his fretting scheme, and the early significance of the method for setting up tempered tunings. A new study of the ancient uses of geometry and arithmetic has been necessary. This shows that Gerle’s scheme had remarkably ingenious theoretical features, which Dowland later modified in a rather expedient way. It has been possible to explain all the changes and their probable purpose. Some new practical features of early Pythagorean tuning have emerged from studying the old methods. I have also devised some new schemes using the ancient approach. Ideally the present title would include all these other topics more fully, but they would not have been analysed without curiosity about the writings of the more famous man. During the study a probable evolution was seen from one type of tuning to another, rather than a sudden occurrence of the different types often implied by modern classifications. Evolution of this sort arises from solving technical problems, and is normal in scientific work. Artistic developments are often different, as for example with the mysterious emergence of the baroque lute style.

Since a virtually full explanation is claimed, it is appropriate to list previous work. In view of the following technical terms this paragraph may be revisited, but it is useful to include it here, rather than in many footnotes. Ref 2 will be necessary for understanding the following work. Murray Barbour listed the frequency ratios obtained from Dowland’s instructions, and their conversion to cents (Ref 4). He thought the system was mainly Pythagorean, but with a very flat third fret and some lesser problems. Then Eugen Dombois wrote that Gerle’s earlier instructions fitted a ‘sixth comma meantone’ system, rather than equal temperament as previously thought (Ref 5). As a leading performer and teacher his work encouraged modern interest in meantone, but his use of cents and long tables overlooked the theoretical and historical basis. This work was reviewed by Mark Lindley, who also wrote that Dowland had copied Gerle’s scheme but made mistakes (Ref 6). His disappointment was expressed in ‘chaotic, inept and garbled plagiarism’.

Robert Dowland’s collection of pieces in tablature was intended for capable amateurs. A first introduction by John Besard is on lute playing, and includes fingerings. The shorter section by John Dowland concerns strings and tuning. These forewords are similar to present day tutors and selections of music. They are not intended to be a complete explanation of the art. They are for ‘schollers not maisters’, and may be a mixture of simple general explanations, with occasional gems of information. There is often an absence of current practical details, and underlying mental approaches, because at the time of writing this was familiar and unnecessary. Our task four centuries later is careful close reading and analysis.
DOWLAND’S TUNING

‘Of fretting the Lute’ - a summary

John Dowland’s section on fretting had a two-page discussion before giving fret positions. Firstly, he said that he would set frets before altering string tensions. This is also a modern methodical approach, rather than mixing the operations. Then he noted how the number of frets had increased recently from seven to ten. Next he noted two methods of setting fret positions: by the ear, which he said was unreliable and sensitive to confusions; or by wit and reason, which we call science. A short ‘biography’ of Pythagoras followed, and the story about the hammers of different weights in the smith’s workshop was given in lengthy detail. This illustrates the principle of exact simple proportions, and the more familiar string length ratios were mentioned. This looks impressive for a busy composer and player, but we may feel let down when the following recipe for fret positions does not lead directly to some miraculously sweet, or even useful, tunings. It will be seen that to find the real substance of his writing one needs to take account of its context. In short, the simple perfect proportions and intervals remained his chief interest, rather than presenting some new exact scheme.

Keyboard tunings were often given in terms of beats and much later in commas, whereas lutes could employ fret positions, analogous to the use of tablature rather than staves. The quite complicated constructions of fret positions, the resulting frequency ratios and their conversion to cents have been presented in Refs 4 and 6. The frequency ratios for frets 1 to 12 can be listed as:

\[
\begin{array}{cccccccccccc}
\end{array}
\]

The lower numbers give the order in which Dowland set the frets, and not the fret numbers. This may signal the importance of certain intervals and frets, and their interrelations. Refs 4 and 6 also considered some of the reasonable and very poor intervals within the scale. This may indicate without proof that this tuning would be unusable on a keyboard, which uses a scale of twelve independent notes. In order to understand the following new treatment for lutes it will be useful to consult Ref 2, and the other references. If some background seems lacking here it may be found in the earlier work.

New analysis for a lute

This initial analysis will consider Dowland’s scheme as it stands, without reference to any other early systems. For a lute, notes cannot be chosen independently as for a keyboard, since pitches on each string must form unisons, octaves and many other common intervals with all the other strings. As I reasoned in Ref 2, this may only be achieved if the basic scale belongs to the large family of ‘meantones’, where all the notes in a scale are generated from a selected fixed value for the fourth or fifth. This includes the well-known examples of Pythagorean tuning (PYT) with perfect fifths; 1/4 comma meantone (QCM) with perfect major thirds, often simply called ‘meantone’; and equal temperament (ET), which is about 1/11th comma, where most intervals may be acceptable but none perfect. Other temperings are 1/6th comma (SCM), and an extreme 1/3rd comma (TCM) which has perfect minor thirds. Ref 2 gave a diagram, seen below, showing fret positions for all possible values of the fourth, denoted by $\beta$. Each degree of tempering, or value of $\beta$, is a horizontal on the diagram and the five examples are marked. For some exact numbers, Tables 4 and 7 in Ref 2 will be helpful.

Therefore, if Dowland’s fret positions fall on a horizontal they would provide a consistent and potentially useful tuning. This can be tested easily by marking Dowland’s frets as circles on the diagram. The C string of a renaissance lute in G was the reference and the enharmonics indicated by the solid lines are those appropriate to lute music and technique (Ref 2). These are: C, D♭, D, E♭, E, F, G♭, G, A♭, A, B♭, B, C. Some earlier discussions (eg Ref 4) have assumed less useful and misleading enharmonics, such as C♯ and G♯. If the G string is preferred
as a reference, it has the same notes except for F♯ on fret 11. Dowland's scheme can be plotted on the diagram as follows.

On the C string, F is a perfect Pythagorean fourth higher on fret 5 or f, with a frequency ratio of 4/3 or 1.3333, and hence is the simple proportion of a ¼ string length from the nut.

The successive fifths: G on fret 7; D on fret 2; and A on fret 9, are also Pythagorean with simple proportional positions of 1/3, 1/9 and 11/27 from the nut.

However, the E♭ on fret 3 and B♭ on fret 10 lie on a horizontal corresponding to a much lower value for a fourth of around 1.330, which has no practical musical interest and is way beyond PYT off the diagram.

Next, the G♭ on fret 6 lies on a horizontal for a higher fourth of 1.3348, as for ET.

The D♭ on fret 1, and A♭ on fret 8 are equivalent to a meantone with a fourth of 1.336, in SCM.

Finally, the major third E on fret 4 corresponds to a fourth of 1.3375, as for QCM. Ref 4 also suggested this for a B on an 11th fret.

It might have been sufficient simply to show the fret positions scattered over the diagram, but some guiding explanations have been supplied. This analysis demonstrates clearly that the recipe, as it stands, is completely unusable on a lute. There is a rough centre of gravity or average tempering around ET to SCM, but the spread is large.

Initial technical view

We need to ask what the master was doing and intended. A simplistic technical answer might be a rigorous learned approach, which went awry and may not have been used on his own lute. The four perfect fifths follow the prelude on the smith’s hammers, and could be a reasonable starting point. However, the strange position and frequency ratio for the E♭ on fret 3, which is even flatter than a minor third in PYT of 32/27 or 1.185, is initially perplexing.

Fret 4 at 1.251 is almost exactly a perfect major third of 5/4 or 1.250. This has some theoretical soundness, but it immediately strikes one as incompatible with the four perfect fourths generated on surrounding open strings, by frets 5 and 7. The lute is a demonstration of fitting four equal but imperfect fourths and a third into two octaves. Furthermore, this ‘perfect’ third was obtained as a simple midpoint between the Pythagorean fret 5 and the peculiar fret 3. Fretting schemes of earlier writers had used simple linear interpolation, which has no elementary theoretical basis.

This may have led here to the very flat fret 3, in an attempt to contrive a ‘perfect’ fret 4 from the Pythagorean fret 5. This view is supported by Dowland’s use of the words ‘perfect ditone’, perhaps with some satisfaction and relish, to describe fret 4, while none of the other perfect frets received the adjective.

No explanation was given for the unusual ratio of 33/31 for fret 1, or the need to use it to derive 33/28 for the poor fret 3. The G♭ on fret 6 is also a midpoint between fret 5 and fret 7, but with a fair result. The recipe could be summarized as four perfect fifths from F up to A, a perfect fifth based on a reasonable D♭, a perfect fifth based on an E♭ with a poor value and derivation, and a perfect fifth based on a low inconsistent but ‘perfect’ E. All frets above fret 6 are perfect fifths on lower notes, as seen from the frequency ratios.

Historical context

We also need to view Dowland’s method in historical context. In his time, general theoretical accounts of tuning in England were still based on many successive Pythagorean fifths, as in Morley’s work noted in Ref 4. We can recall that before the recent arrival of electronic meters all tuning began by setting up perfect intervals such as fifths and thirds, which could be adjusted or tempered according to some ideas and rules. Prescriptions of almost final tunings, as expected nowadays for lute frets, were not normal for the ancients, or even possible, as will become clear.

The problems of tuning were addressed by early theorists who may have had little interest in playing, and also by players who wanted practical systems but also wrote instructions. Some surviving schemes may not have been used, and more useful ones may have been lost. Lute players would have needed a final position of their frets in some effective meantone. A simple
scheme could use perfect fifths for frets 5, 7, 2, 10, which would need relatively little adjustment, and maybe even frets 3 and 9. Then some inventive description might be provided for the other more sensitive frets. It will be seen below that most early schemes would require final adjustments, but our inquiry needs to begin with the written instructions, before considering the probable direction and size of the changes. It is also possible that from one fretting recipe early writers would have produced several different tunings. Changes could even be made between and within pieces. This should be born in mind when, for convenience, later discussion may imply a composer’s single fixed tuning.

Gafurius (1496) may have been the first to write about tempering fifths, then Aron (1523) described QCM, followed by many other varieties of meantone. Meanwhile, for an organ Grammateus (1518) formed five pairs of equal semitones about the ‘black’ notes. For a lute, Bermudo (1555) followed Grammateus, and approximated an ET system using complicated linear divisions along the string. More practically, Ganassi (1543) obtained roughly equal semitones by linear divisions, perhaps giving some indication of a meantone. Then Galilei (1581) presented the 18/17 rule for twelve equal semitones of ET, related to the (18/17) x (17/16) division of a 9/8 tone. This brief survey has used some technical labels, but the early theorists usually simply presented a scheme with no title or explanation. The labels have mostly been added by modern writers, and we need to take care of being preconditioned. The above schemes, and many more, were an active European research project into a baffling technical problem. The degree of obsession was not unlike the alchemists’ searches, or modern particle physics. Apart from historical interests, this has subsided into the acceptance that no perfect tuning solution is possible. Today there is a coexistence of almost ET keyboards, harps and guitars; natural tuning in voices and bowed strings; traces of the harmonic series in the brass; and particular biases in woodwinds and percussion. The hundreds of intricate tunings in Indian classical music tend to use only one melody string, but had great trouble with the recent arrival of the harmonium keyboard.

However, the most important work for understanding Dowland’s fretting is that of Hans Gerle, a composer, player, theorist and maker, and his early scheme of 1532 was discussed in Ref 5. In Ref 1 Dowland explicitly cited Gerle’s book in relation to the number of frets increasing from 7 to 10 over the intervening 68 years, but did not say he was actually using the earlier scheme. Lindley noted that Dowland had copied Gerle, and gave a description of a problem on fret 3 (Ref 6). In the following new analysis it will be possible to draw many further technical and historical conclusions, and make fuller comparisons using the diagram. In addition to copying, Dowland can be shown to have actively altered Gerle’s scheme to suit his expectations and needs, and he may even have used it. Maybe they both had another common source, or other people did the work, but this does not affect the essential matter. Before starting the analysis it is necessary to discuss the geometric methods of the ancients, their general approach and underlying mentality.

GEOMETRIC CONSTRUCTIONS

1. For millennia the usual method of providing sizes for making anything was a geometric construction, in words or a drawing, as discussed in Ref 7. For us, perhaps the most familiar use is the concept of ideal proportions as used for sculptures, buildings, and even lute designs. This generally involved comparisons, or ratios, of simple whole numbers, or integers. The criteria were often aesthetic taste and so varied over time and locality, rather than remaining fixed. Sometimes there was religious and philosophical significance, and more constancy.

2. Closely related to this in ancient times are what we would call scientific descriptions, which define some fundamental properties of space and nature. The chief examples occur in pure geometry and physics. An example here is the Pythagorean frequency ratios, which can be defined by geometric construction. Dowland started with finding the midpoint of a string for the
octave, for which the compasses would need to be large, and then he made the divisions of 3/4, 2/3, etc.

3. Even when no ideal or scientific features seem to be involved, geometric construction and divisions according to ratios are present. The ancients were not working with a standard ruler, inscribed with very small divisions, and calibrated against a reference bar. They appear to have made up piecemeal economical systems sufficient for each specific task. The writers on frets may seem to use this method even when the fret positions are not based on any rigorous musical theory. Examples are the linear interpolations, where Dowland took a midpoint to place frets 4 and 6. He also casually ordered a division into eleven equal parts, which led to the unusual 33. He may have had no method for this except by a trial, ideally needing smaller compasses. The interpolations could also be just a convenient way of describing known positions.

4. An important aspect of the ancient method is very unfamiliar to us, because we moderns are accustomed to a safety net of standard rulers, maps, etc. It can be seen best by an example, such as the linear division into eleven parts. This seems arduous to us and also excessively specific. It is not so easy to set up and adapt as say 2, 3, 4, 6 or 12 parts. The final point will be clearer if we resort to modern measurements. Each of the 11 parts in Dowland’s instructions would be about 18mm on a typical string length of 600mm, since the divided length is 1/3rd between the nut and fret 7 (600/33 =18). Then there is a further division into 3 parts of about 6mm, made on two of the eleven parts. All this work in effect just produces a very short ruler in a fixed position with a finest division of 6mm. This may indicate that the early writers had a good prior knowledge of their desired fret positions, either by some theory or trial, and that these would be described well enough by their chosen divisions, which had a very course resolution. The measuring device and the object are tightly linked together in a single entity. Although this has some elegance it is also potentially difficult to set up, very restrictive, and modifications would not be easy.

5 A further important consequence is that it would be almost impossible to describe very small deviations from simple constructions such as Pythagorean frets. This would need divisions into perhaps 1mm parts, or a modern ruler. When the early theorists made their first Pythagorean intervals, which change only very slightly over the whole range of temperaments, as seen in the diagram, we should not be tempted to weight their systems towards PYT. They would have made small adjustments, but had no way of describing this with their large divisions, or before an exact theory of meantone. Our main focus should be on their intervening frets, which vary greatly between the various writers, and are able to define more strongly their different tuning systems.

6. A remarkable feature of the ancient instructions on frets is the general absence of written numerals and ratios. The ancients evidently thought it was sufficient to write about these matters in a purely geometric way. They may have thought in these terms and even performed intricate calculations, which seems unnatural to us. This feature has been bypassed by the modern immediate conversions of old schemes to decimals, mm and cents.

7. From the previous points, some of the complicated divisions could simply be viewed as a method of measurement, which we might later convert to numbers etc. However, the resulting precise ratios and integers ratios should also draw our attention because they may have represented some significant underlying theoretical musical relations. As seen below, I have found this to be the case with the numbers 11, 33 and 99. These strange divisions then become promoted towards the simpler scientific cases.

8. For the divisions of the string length it is likely that the ancients used a trial and error method. Their tools were compasses and a strip of wood between the nut and bridge. Bisection and trisection can use circles, and five or more parts can use parallelograms. All these constructions require a large area around the line, which would be very inconvenient. This is not mentioned in the sources examined below and some of the methods may not have been known. However, trial and error division becomes difficult when the number of parts is a large prime number, and extremely large numbers may also be less likely to have theoretical meaning.
It needs to be remembered that this 16th century work is part of the renaissance of classical learning, but the scientific enlightenment of Newton et al had not yet arrived. On one extreme the unusual numbers are not numerology or superstition, but on the other hand it is not sufficient to translate all the fret constructions into exact modern equivalents. These simple modern conversions can even be misleading, as hinted above and shown later.

There is an interesting necessary exception to these approaches. When Talbot wanted to describe in detail, rather than make, the baroque wind instruments of his time he had to use inches and fractions. In contrast, a clergyman in the last century could still write a book with instructions for making a complete pipe organ, using only the old method of proportions.

For us, a clear and straightforward method of providing usable final fret positions could be an accurate drawing, which would be easy to scale for different lute sizes. Fully dimensioned technical drawings, sometimes derived from equations, are a modern practice. However, scaling and copying would also have been possible for the ancients, and might have been an accepted method for personal use, if not for publication. The diagram, in Ref 2 and below, shows fret positions for any meantone tuning, and it could be used reliably by anyone.

GERLE'S FRETTING SCHEME

This scheme predates Dowland’s writing by seven decades. The style of the instructions is almost identical and includes a strip of wood between nut and bridge, dividing with compasses, and pricking positions. The frequency ratios that I have calculated from the constructions can be listed as:

\[
1, \frac{33}{31}, \frac{9}{8}, \frac{99}{83}, \frac{792}{629}, \frac{4}{3}, \frac{24}{17}, \frac{3}{2}, \frac{8}{5}, (\frac{27}{16}, \frac{297}{166}, \frac{1188}{629}), 2.
\]

The lower numbers give the order of construction, which is exactly the same as Dowland’s case. Gerle’s highest fret was 7, but he also said a fret 8 could be made closer to fret 7 than is fret 6.

An equal spacing is given above and this produces a perfect minor sixth of 8/5. For frets 9 to 11, I have added Pythagorean fifths on frets 2 to 4, analogous to Dowland’s instructions described above. Gerle used fret numbers, whereas Dowland later used letters and made some mistakes, maybe while copying but also for his own fret 10. These are easily corrected, and have no significance other than indicating a lack of concentration. Gerle did not name the interval at each fret, but Dowland later added them.

It is seen that Dowland’s scheme for the first 7 frets is just Gerle’s, except for frets 3 and 4, which raised problems above. These two important frets that differ from Dowland’s, and also fret 8, are shown on the diagram as open diamonds. Added frets 10 and 11 are shown in brackets. His other frets, which Dowland copied, are provided by the circles.

Frets 1, 3, 4 can now be addressed in new illuminating detail. Gerle’s construction puts frets 1 and 3 almost exactly on the horizontal for SCM. At a time long before exact formulations of dividing the comma in tempering, this may have been achieved by choosing a suitable ratio for a diatonic semitone on fret 1 to suit his music. He could also have tuned fret 1 on the third course to the octaves produced by fret 3 on the first and sixth courses. However, this might only be needed as a check, because a more theoretical method can be deduced from exact ratios. The early theorists deployed a wide range of fractions for a semitone. For example, Ganassi’s scheme, as seen below, was equivalent to a string of semitones involving the five ratios: \(\frac{16}{15}; \frac{17}{16}; \frac{18}{17}; \frac{19}{18}; \frac{20}{19}\). The first is the natural or ‘just’ diatonic semitone, the third is close to ET, the last is close to PYT, and 15/14 would be close to QCM. This shows that the value for an intermediate tuning like SCM would lie between 15/14 and 18/17, or better still between 16/15 and 17/16. The simplest ratio is 33/31, between 32/30 and 34/32. I have not found any other mention of this interval ratio or its derivation, and Barbour’s standard reference work has no entry on Gerle. We now have an underlying theoretical reason why Gerle constructed a division between the Pythagorean fret 7 and the nut, which is a \(\frac{1}{3}\text{rd}\) string length, into eleven
parts, and then put fret 1 two parts (2/33) from the nut, giving a frequency ratio of 33/31. This was his first position after the octave and fifth, so it was highly important for him, but there may have been some even earlier mathematical origin.

Next he described frets 2 and 5 by Pythagorean approximations, using ninth and quarter divisions of the string. Then he put fret 6 at the simple midpoint between 5 and 7, which places it on the horizontal for ET. This fret can be needed for both enharmonics, and would require only slight adjustment.

For fret 3 Gerle made a further division of the (2/33) length between the nut and fret 1, into three parts, each of (2/99). He then took five of these parts (10/99), using compasses, and put them above fret 1. This positioned fret 3 at (16/99) from the nut, giving a frequency ratio of 99/83. This looks even more curious and inscrutable than fret 1, but after considerable thought there are three good reasons. Practically, the B♭ octaves from fret 1, and other chords, would lead to a position for fret 3, and then the simplest subdivision could be sought. However, several more theoretical methods can be deduced from exact ratios. One route could take extreme values of a tone in PYT as 9/8, and in QCM as very close to 19/17, giving an intermediate SCM value of 93/83. This can be seen from the fractions 90/80, 93/83, 95/85. A tone above 33/31 for fret 1 is therefore 99/83, exactly as Gerle constructed fret 3. Using the larger simple 9/8 would have pushed fret 3 onto a more tempered horizontal. This would have been much easier, but Gerle clearly made an effort to reinforce his definition of the tempered SCM system. This is a considerable early achievement, far beyond a mere practical player who stumbled on this ‘simple’ system, as suggested by some modern writers. The key mathematical point is that 93 is 3 times 31, which allows a simple further division of the 33 parts into 99 parts. This reason is quite direct and simple, and may have been used by Gerle, but another can confirm the result. Here the notes F, B♭, E♭, A♭, D♭, obtained as fourths successively further away from C, become progressively more tempered. This means that significant ancient details can strongly identify the tuning scheme, as with Gerle’s fret 1. In contrast, whole tones such as D♭ to E♭ are only two fifths apart, and vary only slightly, similar to fret 2. The midway semitone of 18/17 gives a tone of (18/17)², which can be expressed as 93/83. This can be seen from the fractions 93/88, 90/85, 88/83, since (18/17)² = (90/85)² = (93/88)(88/83) = 93/83. An exact calculation of the SCM tone from Ref 2 is 1.1205. This is identical with 93/83, and compares with 1.1211 for (18/17)². The great sensitivity in selecting useful divisions along the string can be demonstrated by Dowland’s fret 3, which is explained in detail below. He moved it half a 99th division, or about 3mm. This lowered the E♭ ratio from Gerle’s 1.193 down to 1.179, which is a whole comma and way beyond PYT.

Fret 4 was set by Gerle as just between 3 and 5. Previously he had set fret 6 as midpoint, and this may be the intention for fret 4. This would place it on a horizontal for ET, midway between the SCM and PYT systems. Perhaps a suitable ratio for a chromatic semitone could have been added to fret 3 for a less tempered E of about 1.255 in SCM, rather than 1.259 for ET. The adjustment needed would be only about 2mm. This is comparable with the amount needed for the less sensitive Pythagorean fret 2, but more than the 1mm for frets 5 and 7. The simplest ratio for the chromatic semitone would be 20/19, as can be seen on the diagram, but this would entail another lengthy construction.

This analysis demonstrates clearly that Gerle’s scheme is what was later known as ‘sixth comma meantone’ or SCM. Equally importantly, we can see for the first time the theoretical basis of his scheme, and this can provide valuable insight into the ancient methods and intentions.

Although Gerle did not need higher frets their initial positions would not be so crucial. They are used for only one or two treble courses, rather than all six courses. The amount of adjustment required would be about half that for the lower frets, but divisions for construction would also need halving. It was shown above that the initial ancient Pythagorean frets should not be taken as biasing a scheme towards ‘zero comma’ PYT. A similar subtle effect can be seen with the
additional perfect fifth constructions for frets 8 to 11, and this also applies to Dowland’s added higher frets. If a fretted flat enharmonic, such as D♭, on a tempered horizontal is raised by a perfect 3/2, then the higher fret A♮ is effectively pushed into a system with an even higher degree of tempering. In order to remain in the same system the narrower tempered fifth would be needed. On the diagram these higher frets are placed on the lower altered horizontal, for correctness. However, one should probably not view them as intending a system with greater tempering. Alternatively, they might be balanced against the initial perfect fifths. The opposite effect occurs with a sharp type of enharmonic such as E, from which a perfect fifth B on fret 11 belongs to a less tempered system.

Initially it was appropriate to analyse Dowland’s scheme as an isolated entity, but now we must reassess it in the context of Gerle’s earlier scheme. Before that it is necessary to present some important new aspects of Gerle’ work and Pythagorean constructions. Ganassi’s later method, is also of interest since it is a contrast with Gerle’s instructions, but has several relations with Dowland’s.

New conclusions on Gerle
Gerle’s construction clearly has numerical values of SCM. However, a detailed study of the old methods has led me to the conclusion that it would not have been easy, or even possible, for the ancients to provide constructions for other degrees of tempering. Gerle himself probably initially intended such a well-defined degree of tempering, since frets 1 and 3 required unusual effort. Later, he and many other following theorists and players may really have wanted to fine tune to say 1/8th or 1/5th comma effective tempering, but the appropriate divisions may not have been mathematically available. Initially, this further study might have been a small qualification for the sake of rigor, but it has led to further understanding of ancient tuning instructions.

The matter can be investigated by listing the numerators of the frequency ratios for fret 1, from ET down to SCM as: \(72, 71, 70, 69, 68, 67, 66\). It is these numerators that determine the divisions, and the denominators of the ratios are 4 smaller than the numerators. The first 72 needs 18 parts, or 6 from fret 7, or 2 from fret 2. Then 71 needs 71 parts; 70 needs 35 parts or 5 times 7; 69 needs 23 from fret 7 and is about 1/8th comma; 68 needs 17 parts from fret 5 and is about 1/7th comma; 67 needs 67 parts; and 66 needs 33 parts as used by Gerle. Therefore, ET was possible, the 1/7th comma type could have been set but needed a different finer division, the 1/8th comma type is more difficult, and the others are even worse. Further ratios are 52/49, and 50/47 near SCM, but these are not simple. This indicates that a system with less tempering than SCM might only have been practicable by making later adjustments to an initial construction for SCM, or ET.

Similarly, between SCM and QCM the numerators are 65, 64, 63, 62, 61, 60. The only reasonable divisions are for 64, which needs 4 parts from fret 5 for about 1/5th comma; or 60 needing 5 parts from fret 7 for about QCM. Further ratios are 47/44, 49/46, 78/73, 79/74, but all these are difficult. This might suggest that if the ancients had wanted a more tempered system it would be easier to define than SCM. Fret 1 for the 1/5th comma type could be 1/16 from the nut with a frequency ratio of 16/15, and for the QCM type the corresponding ratios are 1/15 and 15/14. However, this analysis has been for fret 1 only, and constructions would then have been needed for fret 3. After more experimental maths (very respectable in number theory, and hence the many famous conjectures) the best I can find for fret 3 is 47/42 on the fret 1 ratio of 16/15 giving 1.194 in 1/5th comma, and 19/17 on 15/14 giving 1.196 in QCM. Both of these require new difficult divisions rather than building on fret 1 as in Gerle’s scheme. In essence, the whole tone above fret 1 needs to be narrowed slightly for more tempered systems in order to give a correct fret 3.

All this suggests that Gerle’s scheme might well have been considered unique over the whole range of practical meantones. The ancients may have known no other moderately complicated ratios and divisions that could provide intermediate degrees of tempering. In contrast, later
methods of dividing the comma, and completely general formulations, can specify any meantone, as in Ref 2 and on the diagram. Gerle’s scheme would therefore have been seen as the best, or only, way to construct the initial set-up for any tempered meantone type of lute tuning. It could then have been adjusted to either side of SCM, even reaching to ET or QCM, but maybe centred on SCM, by intention and default. It is significant that SCM is centrally placed between the extremes of PYT and TCM, and also between the practical musical systems ET and QCM. It is also seen that ET lies between the older PYT and the later SCM, so that music could be said to have jumped a fence and then returned to sit on it for the last two or three centuries. Gerle’s very early scheme of SCM would have been a bold modification of the older purely Pythagorean system to give a distinctly new general type of system that was tempered, as much as an intention to dictate an exact 1/6th comma tempering.

This new idea needs checking by further searches for suitable ratios that could discriminate other degrees of temper. For integers up to 200 there are 40,000 ratios, with maybe a tenth of potential interest, and needing various approaches since there is no simple criterion. A very direct and intelligible method could use modern mm. The positions of the first five frets on a 600mm string for the five systems on the diagram can be tabulated as:

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<thead>
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<th>Fret</th>
<th>PYT</th>
<th>ET</th>
<th>SCM</th>
<th>QCM</th>
<th>TCM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30</td>
<td>33</td>
<td>36</td>
<td>39</td>
<td>42</td>
</tr>
<tr>
<td>2</td>
<td>67</td>
<td>65</td>
<td>64</td>
<td>63</td>
<td>62</td>
</tr>
<tr>
<td>3</td>
<td>94</td>
<td>95</td>
<td>97</td>
<td>98</td>
<td>100</td>
</tr>
<tr>
<td>4</td>
<td>125</td>
<td>124</td>
<td>122</td>
<td>120</td>
<td>120</td>
</tr>
<tr>
<td>5</td>
<td>150</td>
<td>150</td>
<td>151</td>
<td>151</td>
<td>152</td>
</tr>
</tbody>
</table>

These values from exact theory come from Table 7 in Ref 2, and for clarity are rounded to the nearest mm. The variation of position with tempering can be seen for each fret. The sensitivity is greatest for fret 1, and then decreases for frets 3, 2, 4, 5. Next these values can be given in 3mm units, which is about half of Gerle’s smallest division:

<table>
<thead>
<tr>
<th>Fret</th>
<th>PYT</th>
<th>ET</th>
<th>SCM</th>
<th>QCM</th>
<th>TCM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
</tr>
<tr>
<td>2</td>
<td>22</td>
<td>22</td>
<td>21</td>
<td>21</td>
<td>21</td>
</tr>
<tr>
<td>3</td>
<td>31</td>
<td>31½</td>
<td>32</td>
<td>32½</td>
<td>33</td>
</tr>
<tr>
<td>4</td>
<td>42</td>
<td>41</td>
<td>41</td>
<td>40</td>
<td>40</td>
</tr>
<tr>
<td>5</td>
<td>50</td>
<td>50</td>
<td>50</td>
<td>50</td>
<td>51</td>
</tr>
</tbody>
</table>

This shows that frets 1 and 3 can be distinguished in all five systems, whereas frets 2, 4, 5 vary much less, and would require much finer divisions. Even fret 3 really needs 1½mm division, which gives the frequency ratios on frets 1 and 3 in the five systems as:

<table>
<thead>
<tr>
<th>Fret</th>
<th>PYT</th>
<th>ET</th>
<th>SCM</th>
<th>QCM</th>
<th>TCM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>396/376</td>
<td>396/374</td>
<td>396/372</td>
<td>396/370</td>
<td>396/368</td>
</tr>
<tr>
<td>3</td>
<td>396/334</td>
<td>396/333</td>
<td>396/332</td>
<td>396/331</td>
<td>396/330</td>
</tr>
</tbody>
</table>

The fret positions would be 20/396 etc from the nut. Intermediate tunings could be defined for fret 1 but not for 3. Several positions can still use larger divisions and the sizes in mm are listed underneath the ratios as:

<table>
<thead>
<tr>
<th>Fret</th>
<th>PYT</th>
<th>ET</th>
<th>SCM</th>
<th>QCM</th>
<th>TCM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>99/94</td>
<td>18/17</td>
<td>33/31</td>
<td>198/185</td>
<td>99/92</td>
</tr>
<tr>
<td></td>
<td>198/167</td>
<td>132/111</td>
<td>99/83</td>
<td>396/331</td>
<td>6/5</td>
</tr>
<tr>
<td>mm</td>
<td>6 &amp; 3</td>
<td>33 &amp; 4½</td>
<td>18 &amp; 6</td>
<td>3 &amp; 1½</td>
<td>6 &amp; 100</td>
</tr>
</tbody>
</table>

This demonstrates practically how complex ratios and finer divisions can be needed to define even the simple systems of PYT, ET, SCM, QCM and TCM on the most sensitive frets.
However, the initial approach was more subtle and looked for special cases that could use larger divisions. The form of fraction for fret 1 can be generalized as \((14m + n)/(13m + n)\) where \(m\) is given values of 1, 2 etc, and \(n\) takes values from 1 to 4m. This accounts for all fractions in the range between ET and QCM, but it is still necessary to write out all possibilities and assess whether any are useful. The criteria are: fractions do not need a preset exact value; numerators should not be large prime numbers requiring difficult divisions; numerators may be products of relatively few moderate primes; and extra fractions close to the principal five systems are not the main interest. The whole exercise took many pages but the better candidates can be listed as:

**QCM**
15/14, 76/71, 77/72, 91/85, 107/100, 108/101, 121/113, 136/127

**Intermediate**
16/15, 63/59, 95/89, 110/103, 112/105, 143/134

**SCM**
33/31, 49/46, 50/47, 65/61, 114/107

**Intermediate**
17/16, 52/49, 84/79, 120/113, 121/114, 154/145

**ET**
18/17, 35/33, 88/83, 125/118, 143/135, 160/151

The underlining indicates some promising numerators with small prime factors. At various stages one may be tempted to tidy up the awkward numbers, but 1 in 150 is equivalent to 4mm. After a long process, amounting to a proof by inspection and exhaustion, no candidates appeared to approach the suitability of 15/14 for QCM, 33/31 for SCM, 18/17 for ET, and the intermediate values of 16/15 and perhaps even 17/16. The 33/31 ratio can be regarded as a second tier with \(m = 2\), whereas the other simpler ratios have \(m = 1\).

A major topic in number theory is ‘continued fractions’, where any real number, including rationals or fractions, decimals, irrationals including square roots, \(\pi\) and \(e\), can be expressed as:

\[
a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\ldots}}} \]  

This would be valuable here if an exact frequency ratio needed to be expressed accurately as a fraction, using successive \(a_k\) to obtain better convergent values. However, in the fretting problem a more important requirement is moderately simple fractions within a range. Also, continued fractions do not generate all possible approximations, but we need to have all fractions for an assessment of suitability.

A derivation of the \(a_k\) can be seen from the simple example of say 1.060. Here, \(a_1 = 1\); then \(1/(0.060) = 16.666\ldots\) so that \(a_2 = 16\); then \(1/(0.666\ldots) = 1.50\) giving \(a_3 = 1\); then \(1/(0.50) = 2.0\) and \(a_4 = 2\). The \(a_k\) integers are written as \([1, 16, 1, 2]\), and then successive approximations, known as convergents, can be calculated as 1, 17/16, 18/17 and 53/50. Here 53/50 is clearly a final check on 106/100, but infinite series are more usual. This example can also illustrate rounding errors. Truncation at 0.666… would give \(a_4 = 1\), then \(a_5 = 1\), and a very large \(a_6\), indicating an end point. This produces a further convergent, 35/33.

If the method is applied to fret 1 ratios calculated by exact theory, then 33/31 for SCM shines out once again as a remarkable number. The successive convergents for exact SCM are 16/15, 17/16, 33/31, 2096/1969. These can be compared to the less convergent ET with 17/16, 18/17, 89/84, 463/437 etc, and to QCM with 15/14, 46/43, 61/57, 107/100. Also for QCM a whole tone is exactly \(\sqrt[5]{(5/4)}\), and the square root can be expressed as a ‘periodic continued fraction’ with convergents 9/8, 19/17, 161/144. The second 19/17 is very accurate but has awkward divisions.
A new construction for $1/8$th comma meantone

Yet another approach is to start from fret 3 for a minor third. This further use of the subtle method yields a very nice and practical result. Exact frequency ratios for the extremes of PYT and TCM are $32/27$ and $6/5$, which leads to the following intermediate values:

<table>
<thead>
<tr>
<th>Fret</th>
<th>PYT</th>
<th>ET</th>
<th>ECM</th>
<th>SCM</th>
<th>QCM</th>
<th>TCM</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>128/108</td>
<td>126/106</td>
<td>125/105</td>
<td>124/104</td>
<td>122/102</td>
<td>120/100</td>
</tr>
</tbody>
</table>

Although the numerators for ET, SCM and QCM are difficult and no improvement, it is seen that new intermediate cases are possible. The area of most interest is between ET and SCM with a ratio of $125/105$ or $25/21$, but the other intermediate numerators are difficult or unnecessary. The ratio of $25/21$ is a touch of serendipity, even better than $33/31$ for fret 1. Now we can search back for suitable ratios on fret 1, and candidates found above were $17/16$ and $120/113$. The large prime numerator 17 would be more awkward than 11 and has no relation with fret 3, but 120 has many small factors, and one in common with 25. The factorizations are $2 \times 2 \times 2 \times 3 \times 5$ and $5 \times 5$.

These new ratios are not in Ref. 4, and it is probable that the ancients did not find them. This may be a significant discovery, but 500 years late.

This intermediate case is almost exactly $1/8$th comma meantone. Using details from Ref. 2, the frequency ratio for $D^b$ is 1.06195, which is 0.132 comma, and for $E^b$ the ratio is 1.1905 which is 0.120 comma. In the table above this case is labelled as ECM, and the fret positions are shown on the diagram as solid diamonds. In practice a first division would have 5 parts of 120mm, and then a further division into 5 parts of 24mm next to the nut would give fret 3 at 4 of these parts from the nut. This is 96mm, which exactly fills the gap in a table of the previous section. For fret 1, a 3 part division of 120mm next to the nut gives 40mm, and the further 8 parts are 5mm. Fret 1 is 7 parts or 35mm from the nut, and is more central than suggested by the rounded numbers in the table. It is remarkable that these large divisions of 24 and 5mm, but with no common factor, can define frets better than the previous impractical 396 parts of 1½mm. This is the beauty of the ancient method at its best, but there are very few possibilities.

The relation between frets 1 and 3 in this ECM is different from Gerle’s original. In his SCM fret 1 was relatively simple, and fret 3 was an added complication. In contrast, for this new ECM the fret 1 ratio is quite complex apart from the numerator, but fret 3 is largely independent and even simpler than Gerle’s fret 1. The main reason for trying this approach was that the derivation of Gerle’s ratios from intermediate semitones and tones began to look rather convoluted, and it was seen that fret 3 could use two exact limits for a minor third.

In this ECM the difference of the sensitive frets 1 and 3 from exact SCM and ET is only about 1mm. This is comparable with the ancient accuracy in defining the less variable frets 4 and 5. Therefore, even if the ancients had found frets 1 and 3 for ECM, and maybe other rare intermediate cases, the accuracy could have been disproportionate and unnecessary. However, it would be very significant if an ancient source contained such alternatives to Gerle.

For a final new use of continued fractions, the above table shows that fret 3 in SCM has a difficult ratio of 31/26. The extremes of PYT and TCM can also be written exactly as 162/135 and 160/135, giving an alternative mean of 161/135 for SCM. Gerle’s original 99/83 is very close in value, but is not easily derived from 161/135. However, their continued fractions can be written as $161/135 = [1, 5, 5, 5] = 1.1926$, and $99/83 = [1, 5, 5, 3] = 1.1928$. This shows how very different looking but almost equal fractions could be generated by changing the final $a_k$. In general, the denominator of convergent $k$ will be $1 + a_k \cdot a_{k-1}$. The fractions have no other relation except the previous convergents $31/26 = [1, 5, 5] = 1.1923$, and $6/5 = [1, 5] = 1.20$. Just for interest, some fractions with other $a_4$ are 37/31, 68/57, 130/109, 192/161, 223/187. In this case they are little practical use, but the method could be helpful with other ratios. One could generate many useful ratios within a specified deviation from some exact value. This new method could have uses in other practical areas. Number theory is the only traditional branch of
maths generally unvisited by physicists, so I did a course for curiosity rather than practical use. Perhaps the only direct application is in modern security codes. Two enormous primes are multiplied, but these original factors cannot presently be recovered without prohibitively long trials. It is not clear whether maths or security and commerce will be the final winner.

General conclusions on ancient methods

These new findings depend on understanding the restrictions of the ancient constructions. Our modern reference systems of decimal points and finely divided rulers were not in ancient use. Previous modern conversions of Gerle’s fret positions into cents carried the tacit assumption that alternative systems could have been constructed, but this may not have been possible, or expected. It is amazing that from the whole range of meantones lying between PYT and TCM on the diagram, the early theorists might have known at most four using Gerle’s approach: SCM, ET, QCM and TCM. The final two are unlikely extremes for practical use and could use alternative methods starting from perfect thirds. It was noted in Ref 2 that while QCM can sound well on keyboards, this large tempering may become impractical on a lute due to the alternate very narrow and wide spaces between frets, which can be seen in the above tables. It is also worth observing that for keyboards the early types of meantone had very strong tempering such as QCM and TCM, and the milder types such as SCM, ECM and ET only appeared much later (eg Ref 4). This does not overrule or invalidate the early fretting constructions for milder tempering of lutes, or their use by modern players of lutes.

Conclusions from these new analyses may apply to many other constructions given by early theorists, and also to the work of modern writers. Perhaps some early schemes have been classified too hastily as ET by modern opinion, analogous to the strict SCM for Gerle, and this will need future study. Bermudo’s scheme is notably complicated, but I have deliberately avoided it for now, in order to keep below 50 pages. With regard to the musical aspects, it has been thought that some composers, including Luis Milan, Schlick, and Gerle himself, had in mind a precise SCM rather than some other tempering. It may be useful to re-examine these uses. It is ironic that 40 years ago Dombois demonstrated that Gerle’s scheme was not ET, which had been simply assumed by Mitchell and Meylan (Ref 5). Now it can be seen that both views had some validity, but from incomplete analyses. For an authentic approach a modern player might set up frets and temperaments by the ancient method of divisions for SCM and ECM followed by fine adjustments. For example, one can see from the diagram that weaker tempering requires frets 1 and 3 to be moved slightly towards the nut. The diagram shows the direction and degree of adjustment needed between any initial set-up and desired tuning, and this guide would be useful even to those who tune by ear and eye.

The above complexity has been necessary, in order to show that the ancients could not construct fret positions at will for any degree of tempering. The next section might be missed on a first reading, but it leads onto new ideas about earlier Pythagorean schemes.

Further details on Gerle

Gerle needed only 7 or 8 frets, but he may have been able to construct good higher frets. A general property of any meantone system is that the semitone from B to C is identical to C to D♭, as seen in Ref 2. The smaller divisions needed for fret 11 can be estimated as (33/31)/2 times the 1/11 part needed for fret 1. The division is not integral but is near to 21 parts, and might be refined. Then fret 9 could be constructed like fret 3, but fret 8 would be difficult like fret 4. The relations also demonstrate another general property of meantone: that the product of frequency ratios for a pair of notes such as D♭ and B, then D and B♭, and up to F♯ and G♭, is equal to 2. Similarly, but more simply, fret 10 for a B♭ could be taken as a perfect fourth above F. Then a simple Pythagorean fret 3 for E♭ with a frequency ratio of 32/27 could be 5/27 from the nut. Later this would have suited Dowland better than his awkward flat E♭ with the related ratio of 33/28, and he could have continued with A♭ and D♭. These successive perfect fourths were part
of Bermudo’s scheme, and the usually simple fifths above C were given complicated constructions.

As seen above, early fretting schemes were often built up by semitones and tones, in addition to a few simple fifths. However, by fret 4 the required ratios and divisions become difficult. This is worth explaining from the analysis in Ref 2. For the scale of notes on the diagram, the semitones from C up to G♭ have alternate values of \((\beta^5/4)\) and \((8/\beta^7)\), and also the same sequence down from C. Here \(\beta\) is the generating fourth for the temperament, as shown on the diagram. This means that each semitone depends on the exact value of \(\beta\), and even more sensitively due to the high power of 5 or 7. Errors would accumulate with each added semitone. Gerle could cleverly approximate frets 1 and 3, but the later exact theory is better for higher frets.

In contrast the whole tones vary as \((2/\beta^2)\). This is less sensitive to the lower power, but requires extremely accurate values of \(\beta\) to define a temperament, which is why the ancients had to take a perfect 4/3, and then later make slight adjustments. Nevertheless, the success above in using exact values for the extreme temperings, together with Gerle’s derivation of 93/83 for a tone, suggests that the ancients might have been able to construct the small variations needed in fret 2. A tone in PYT has the ratio 9/8, and for Gerle’s 1/99th division this becomes 99/88. The next lower ratio 99/89 is smaller than the required 93/83 = 1.1205, and it is also smaller than the tone in QCM of 95/85 = 1.1176. This shows that Gerle’s 99 or 33 parts are not able to define intermediate tempers, as also found above. However, for a division into 66 parts the region of interest would be 66/60, 66/59, 66/58 with values of 1.100, 1.1186 and 1.138. The ancients could have easily constructed a fret 2 with the ratio of 66/59 lying between SCM and QCM. In a similar way, the difficult fret 4 for a major third, or two tones, could have been set from the ratio 99/79 = 1.253, again lying between SCM and QCM. There appear to be no other intermediate ratios available from these divisions. Either the ancients did not find these ratios, or considered the temper too extreme, or the degree of precision unnecessary. Frets 5 and 7 vary too slightly for such methods, but some higher frets might be treated.

A further intriguing effect can be seen from the diagram. The various enharmonic pairs such as D♭ and C♯ cross over on the ET horizontal such that a D♭ in the PYT system appears to have a similar value to C♯ in SCM. Conversely, a D♭ in the SCM system appears to be similar to C♭ in PYT. This might suggest that Gerle’s complex construction for fret 1 could have been avoided by making a series of seven perfect fifth constructions from C to C♯. Rereading the discussion of Dowland in Ref 4, it is said that 33/31 has a value in cents similar to a chromatic semitone for a C♯ in PYT. This is true, but lute music requires a D♭ in order to give A♭, E♭ and B♭ on courses 1, 2, 3, 6. Slips of this sort are common without familiarity with the lute, and G♯ was assumed for fret 8, and F♯ for fret 6, which are less serious. Ref 4 has no mention of Dowland’s derivation from Gerle, nor his original fret 3 and SCM.

Pursuing these paired enharmonics, where a flat in SCM is similar to a sharp in PYT, it can be seen and deduced that it is only exactly true for the symmetric G♭ and F♯, which are six fourths or fifths from C. For this pair \((\beta_p \beta_s)^6 = 32\), where the fourth in PYT is \(\beta_p = 4/3\) or 1.3333, and in SCM \(\beta_s = 1.3360\). This is a special relation and not a simple identity. It is equivalent to using 4.994 as an approximation for 5 in the 81/80 syntonic comma. Further away from this midpoint the approximation of sharps in PYT, for flats in SCM, becomes much poorer. For example, it is clear that the above relation would not allow \((\beta_p^3 \beta_s^9) = (\beta_p^9 \beta_s^3) = 32\) for the E♭. The D♭ from C♯ is good; the A♭ from G♯ is moderate but this fret is not crucial; the E♭ from D♯ is very poor such that the sharp is equal to the flat in TCM; and the B♭ from A♯ is worse. A poor E♭ on fret 3 by this procedure is a relief for this analysis and some posthumous satisfaction for Gerle. The next section contains some further surprising difficulties with the practical accuracy of repeated Pythagorean constructions.
For completeness, the frequency ratios from the repeated perfect fifth constructions are:

1, 2187/2048, 9/8, 19683/16384, 81/64, 4/3, 729/512, 3/2, 6561/4096, 27/16, 59049/32768, 243/128, 2.

The labels underneath indicate the temperaments of the corresponding flats.

The exact theory of meantone also has some bearing on how ET was viewed by the ancients, and still today. Initially, a rough ET could be found with some difficulty by linear divisions, including Bermudo’s scheme, and later by the repeated application of the 18/17 semitone. However, the ancients were concerned by a loss of simple intervals and ratios, as in the diagonal of a square, and the properties of a circle. This fear of irrationality also arose with the theory of equal semitones having a frequency ratio of $2^{1/12} = 1.0595$. From the above relations, the fourth is given by $\beta^{12} = 32$, so this interval and all others are also irrational and ‘impure’. These phobias may linger even today, and the best cure is to view ET as just one special case among an infinite set of meantones, as on the diagram. The complicated fractions formed from increasingly large integers, such as Gerle’s 99/83 and 792/629, were considered by the ancients to be more ideal and acceptable than the accurate irrational numbers.

A final query emerges from the geometric instructions in Gerle’s scheme. To make fret 3 he took 5/3 times the distance of fret 1 from the nut and placed this above fret 1, as explained above. The frequency ratio of a perfect major sixth A is also 5/3, and this is complementary with 6/5 for a perfect minor third E♭. The construction for fret 3 provides a new tempered minor third, so it may seem that there is some abstruse iterative process for obtaining better tempered frets from simple perfect ratios. A full examination is lengthy, but it turns out that this is an isolated coincidence. The condition can be expressed as: $1 - (1 + 5/3)(r - 1)/r = 1/(6/5 - c)$, where r is the frequency ratio (33/31) of fret 1, c is the small but significant improvement (0.0072) to the perfect minor third. This simplified form also shows how one can test other pairs of perfect intervals, and initial frets other than fret 1. Briefly, on fret 1 a value of 16/15 for r leaves fret 3 as 6/5, which is an identity of no practical advantage. The four other complementary pairs are not useful and other typical ratios, such as 15/14, do not give improvements. For other initial frets: on fret 2 the perfect pair of G and an octave C produces an identity for a ratio of 9/8; the pair F and G reproduces 10/9; and on fret 3 G and C give 6/5. These identities are interesting combinations of simple ratios, but no other useful tempering was found from about thirty cases. There is certainly no general method, but one cannot rule out an odd remaining useful case.

New conclusions on Pythagorean constructions

The repeated perfect fifths of the last section produce unwieldy ratios. These bring to light some interesting questions about the ancient methods, and possible practical problems, which can be seen as follows. The first fifth would need a 1/3rd division to give fret 7 or G, but if the simple 1/3rd division is repeated then D has a 9/4 ratio, which is above the midpoint or fret 12. Instead the ancients found fret 2 by prior calculation of the ratio 9/8. This may seem trivial but it provides insight into how the ancients thought since, as we have seen, they generally gave constructions without writing down the ratios and the type of enharmonic. From the 9/8, an A of 81/64 on fret 9 could follow simply, like the G. However, if E on fret 4 is taken from the ratio 81/64 some effort is needed to produce the 1/81 division and a position 17 parts from the nut. A much easier method could make an E above the midpoint and then simply double the sounding length. This might continue for the more complex ratios, and a similar series of fourths could be made.

Several early Pythagorean lute schemes have been presented in Refs 4 and 6. From a particular practical method given by the ancients I have deduced their essential plan from the lengthy detailed instructions. A scale of twelve frets for an open C string was built up from the 9/8 tone as D, E, F♯, by taking new 1/9th parts of the preceding sounding lengths. Then above the perfect
ratio of $3/2$ for G they found A and B in the same way, which gives seven frets. In a similar way, the perfect $4/3$ for F was used to form a lower $9/8$ tone, by taking a $1/8$th part of the $1/4$ sounding length and adding this to give an $E^b$, and then a repeat for $D^b$. Frets for $B^b$ and $A^b$ were obtained in the same way from the octave C, giving another five frets. This is much longer than Gerle’s concise and economical method, and would involve no less than six different $1/9$th divisions, and four $1/8$th parts which are a little easier. This particular anonymous scheme dates from 1560, well after Gerle’s, and the better known scheme by Oronce Fine appeared a little earlier in 1530. Interestingly, the $1/9$th division was made in two $1/3$rd divisions by trial, as I inferred above for other schemes. Also, a further method of scaling the frets used an angled axis. This is essentially a simplified reversal of the parallelogram method, which would have been suitable for making difficult equal divisions, if the theorists had known it. With this general procedure some different enharmonics are possible. For example, a $G^b$ could be taken from the $A^b$, as for the usual lute scale; then a less usual $C^b$ from the $B^b$, and this was actually Fine’s scheme. Also a $G^#$ and $A^#$ could be derived from the $F^#$, but then it can be seen that the necessary $C^#$ cannot be obtained by this early method, and might need to come from the upper B, or from an $F^#$ by the above use of fifths. Then a further $D^#$ could complete the sharps. The practical number of schemes would be six, ranging from five flats to five sharps, but one could continue with unlikely enharmonics such as $F^b$ and $B^b$. Previous discussion of single schemes may have missed these generalizations, and especially the unavailable $C^#$ and $D^#$.

In any practical construction it can be anticipated that errors would accumulate with many compounded intervals, because each fret is the starting point for the next. This would certainly occur for the ‘quick’ scheme I initially suggested, and the error would be increased by the doubled lengths. After perhaps four perfect fifth constructions these errors would begin to creep in for $B$ and $F^b$. Similarly for the perfect fourths, errors would appear by about $D^b$ and $G^b$. The schemes with more sharps, or flats, would grow greater errors. A tuning that required twelve successive fifths, or fourths, could produce a highly chaotic set of positions, and be virtually unrepeatable. The scheme from the last section with five sharps from eleven successive fifths, or a scheme with five flats, would have the greatest error. Some unconventional mixtures of enharmonics could be twice as bad. The enharmonics chosen in the anonymous 1560 scheme would need seven successive fifths and give a smaller error.

The actual method in the 1560 scheme also used a chain of dependent constructions, which could produce increasing errors. However, the errors could be smaller because there were only four successive tones and no doubling, but the constructions would take about twice the effort. The method of building parts of a scale from tones could reduce errors. This corresponds with simple melodic usage, and also with the early Greek theorists. Plato’s Timaeus refers explicitly to the common Pythagorean intervals and filling $4/3$ with two $9/8$ and one $256/243$. This may look like musical theory, from where it most likely derived, but it appears in his ‘soul of the world’, rather than his later ‘sounds’. In his writing he did not approve of the emotional power of music and poetry. This much earlier use of tones to fill larger simple intervals presumably carried through to the early fretting instructions, rather than some later change from successive fifths in order to reduce errors.

These sources of error may have been missed by earlier studies thinking in terms of exact ratios and conversions to modern units rather than the original constructions. As an illustration, the $C^#$ in the last section had an exact frequency ratio $2187/2048$, or $1.0679$ in decimals, which places fret 1 at exactly $38.2$mm from the nut. If seven successive fifths were used and each had a potential error of say $1/4$mm, then frets 1 and 10 might have errors around $2$mm. This arises solely from the practical repetition of a simple rule, but it is comparable with the difficulties in defining the more complicated tempered schemes. The ancient experience would have been even more uncertain because they were proceeding into the dark with no exact idea of the correct pattern on the fingerboard. These errors are analogous to a loss of predictability in
situations such as the multiple bounces of a billiard ball. Until recent decades physics tended to
ignore effects of this type, so the ancients would have been alarmed by disorder appearing from
their revered Pythagorean law. It is likely that any very oddly positioned frets would be
relocated, and human nature would tend to place these frets more centrally. This would bias the
tuning from pure PYT towards ET and beyond. The most affected notes and frets would be
precisely those that strongly identify an effective temperament, as seen from the analysis of
Gerle’s scheme.

These ancient uncertainties would apply mainly to fret instructions, but tuning fifths by ear and
eliminating beats could be far more accurate. Practical tuning methods for keyboards, before
meters, gave instructions for tempering that provided several internal checks. For example, in
QCM the four fifths up to E are equally narrowed to give a perfect third. This is how Aron
presented the first meantone, and not in terms of the comma. The method was extended to other
meantones, but ET with no perfect intervals is more difficult.

All these points indicate that for a lute, ancient Pythagorean fretting would have been far from
simple, and prone to error. The theory has always remained as a benchmark, but it might have
been used less than often thought, and errors could have produced a variety of other usable
tunings. It is customary to note that poor initial schemes can be adjusted, but now one can see
that practical versions of PYT could already be distorted, and ironically useful from the start.

Some modern writers (eg Ref 6) have also doubted whether ancient players used the
Pythagorean schemes, but on the basis that they were inappropriate for the music of the early
16th century. An accurate set up in PYT lies on a horizontal and is therefore self-consistent.
Other modern ears have found a well-tuned PYT quite acceptable and this may reflect
conditioning to major thirds that are already quite sharp in modern ET. However, many of the
intervals are simply very poor, and would mainly be useful where some discords are acceptable
in melodies above a few simple chords, as in medieval music.

It was considered above how some ‘wrong’ enharmonics in PYT might be an approximation for
SCM. The main interest in Gerle’s scheme was D♭ and E♭ on frets 1 and 3, but very interestingly
the corresponding sharps were problems in the early Pythagorean constructions. The two ancient
methods above were confined within the octave, and the C♯ and D♯ were unavailable. If there
were alternative schemes that used the method of successive fifths then the errors would be
larger. From this analysis one can see how Gerle’s scheme would have achieved several features
lacking in the contemporary Pythagorean methods, and may have been devised to overcome the
earlier problems. His constructions involved far fewer operations; there was little reliance on
previous positions that can increase errors; the tempered tuning was more suitable for the
current music; and the important frets 1 and 3 were deft initial inventions. These frets had either
been inaccessible or left to poor compound forms in the Pythagorean schemes. Gerle was
probably aware of problems with fret 1, and this would explain why he gave a special early
construction. Then his use of adding a further tone to define fret 3 would have been a feature from
the earlier methods. The ancient use of linear interpolation, such as a simple midpoint that
has seemed non-theoretical, would have been a further simple way of avoiding erratic
compounded intervals. A midpoint is good for Gerle’s fret 6, and not bad for his fret 4, but
Dowland’s fret 4 was poor. Several new interpolations with some very fortunate results are
given below.

Some details in the evidence can best be understood as an evolution of one scheme to another,
rather than separate types to be discussed with modern units and musical taste. However, it
would not be sensible to force everything into a story, or to neglect uncomfortable evidence. In
discussing this period, whether writings or the music, it should be recalled that the earliest
known material on lutes is from the first decades of the 16th century. The Pythagorean scheme of
1530, and especially that of 1560, were probably the tail end of much earlier work, and aimed at
proving their old method against the newer schemes. Gerle only needed seven frets for his lute,
whereas they gave instructions for all twelve frets. This may indicate a theoretical rather than practical use, and they also made the inviting promise of perfect tuning with no need for later adjustments. Another telling detail is that they ordered complete divisions into nine parts, whereas it saves time to make the second three part division only in the area of interest, often just one part near the nut, as with Gerle and Dowland. It appears that schemes like Gerle’s came to be regarded as superior methods, since they were still being cited many decades later.

Ganassi’s fretting scheme
There are many more old fretting instructions in the references, and the website of the Lute Society of America gives easy access to 16 schemes. One can see that most might be of little direct use, but a few may be reasonably close to a meantone, even though the original ratios are not a theoretically correct form. They could be assessed by the present diagram and ratio methods, and this might be useful future work. As seen above, there were several common features of the old instructions: an expected display of classical and musical learning; the use of geometrical construction; and a claim to offer schemes ready for use. In contrast, Ganassi repeatedly called for adjustments by ear and his instructions are lengthy, but his basic scheme is geometrically very simple, unlike Gerle and the Pythagoreans. It contains interesting symmetries, but also some wayward unusable intervals.

Ganassi’s frequency ratios can be listed as:

\[
\begin{array}{cccccccccccccccc}
1, & 20/19, & 10/9, & 20/17, & 5/4, & 4/3, & 24/17, & 3/2, & 30/19, & 5/3, & 30/17, & 15/8, & 2.
\end{array}
\]

Below each fret the position in exact cm is given for a string of length 60cm. The positions are shown on the diagram as crosses, except where they are the same as Dowland’s circles. Frets 5 and 7 are in Pythagorean positions. Then the length from the nut to fret 5 was divided into five equal parts. This produces a perfect third on fret 4, but this is incompatible with perfect fourths on a lute. Fret 2 gives a minor tone, which is far too low, and frets 1 and 3 are also poor and could only be Pythagorean, or incorrect sharp enharmonics. Fret 6 is just the midpoint between 5 and 7, as above. It is ironic that fret 6, which often presents a dilemma for modern players, is the same in all three schemes, but it is not surprising that the all initial positions of frets 5 and 7 are for perfect intervals. The length from fret 7 to 12 was again neatly divided into five equal parts, 2/3 of the lower 3cm parts, so that frets 8 to 11 are perfect fifths on frets 1 to 4. Poor lower frets therefore produce two perfect ratios for frets 9 and 11. This is a strong contrast with Gerle who defined in detail the more important lower frets, and thereby effectively the whole temperament. Ganassi’s frets could be adjusted to some meantone, but there is little indication of any particular system. There is enormous scatter on the diagram and the shifts would be large, especially for frets 1 to 4.

The scheme would be a very simple way of setting some initial positions. This is a contrast with Pythagorean schemes using constructions that are lengthy but prone to error. The appealing symmetry of Ganassi’s fret positions is related to the particular choice of five perfect intervals. The tuning is very specific to the lute’s fingerboard, and would scarcely come to mind for a general theory. The geometry highlights the octave’s division into two fourths about a central tone, or two tritones. Using an equal spacing for only five frets is poor, even as a rough start. This makes it clear that the simplest division of a full octave into equal parts would give extremely poor tuning. The scheme is sometimes labelled as ‘Just with mean semitones’ (Ref 4), which refers to the presence of several perfect intervals, with some indication of tempering. The later intricacies of just schemes brought an end to searches for perfect tuning. However, they can be a first theoretical step in constructing useful tunings. In effect, this is where I began in Ref 2, and then realized that any system for a lute would need to generate all notes from a fixed value for a fourth, leading to a completely general exact formulation of meantone.
Examining Dowland’s instructions without any earlier context such as Gerle and Ganassi, it is certainly seen that they represent a final practical tuning somewhere between PYT and QCM. This would need to be a meantone with a fixed value for the generating fourth, and lying on a horizontal. The average might be about ET or ECM, depending on the weight given to the insensitive perfect fifths, and the poor frets 3 and 10, which would need considerable sharpening. These are also popular choices nowadays for Dowland’s music, but it should be clear that the analysis has not been massaged into agreement. This may not seem much, but it is more than would be suggested by a simple consistent series of perfect fifths, giving no indication of any tempering, but lengthy and liable to error. It is also better defined and has less scatter than Ganassi’s scheme, and several others.

However, the clear derivation from Gerle’s compact and definite scheme may alter this conclusion. We would like to know why Dowland wrote on this particular scheme, whether he had tried it out, used it or relied on it, why he made some changes, were these carried out practically, and did he find improvements. Some answers can be found from his own words.

Gerle’s book was 68 years old but evidently still available, in some form, and respected. It was demonstrated above that Gerle’s fretting would have been appropriate for many types of meantone, and not just restricted to SCM. This more general use may provide a good explanation for his scheme still being in circulation and useful to Dowland. Other schemes, such as the ECM discovered above, pure PYT, Ganassi’s scheme, something closer to ET like the 18/17 rule, a series of perfect fourths from F to B♭, E♭ etc, or even setting frets by sight and ear, might have suited Dowland, but he did not write on them. This might imply they were, in turn, unknown, lacking a perfect third, too scattered, lacking Pythagorean fifths on C, or unscientific. The plainest interpretation is that Dowland chose Gerle’s scheme because he wanted some general features of its significant tempering, and it was known to be an acceptable start for setting any meantone. Logically, one should not dismiss too hastily the extreme possibility that Dowland wanted tuning close to an exact SCM.

Gerle’s book was brief, concise and practical, so that its style and content would provide a useful template to follow or copy. Dowland referred to Gerle’s many occupations in the terms ‘so he styleth himself’. This may seem derogatory but it actually indicates a simple lack of knowledge of the much earlier musician, which is natural in times when musical activity had less historical perspective. He noted that Gerle’s lute had only 7 frets. He continued that he was born 30 years after the book of 1533 was printed, and that by then lutes had 8 frets. Next he wrote confusingly that an Englishman added 3 frets glued on the body, and a Frenchman added 2 frets more? Finally he said the most desired lutes had 10 frets. This explanation of fret number increasing from 7 to 10 has more significance than general historical interest. It clearly refers to the need for his own additions of frets 8 to 10. However, this is his only mention of Gerle, and he did not say later that the first 7 frets are taken from Gerle. Perhaps the earlier scheme was well known, or Dowland did not consider it necessary to repeat the reference, or he forgot.

Since Dowland’s only reference to Gerle is the increase from 7 to 10 frets, this may have been his main concern and it is well to address this first, before the more interesting frets 3 and 4. His writing on the fret constructions differs slightly in the exact words, some of which may result from translation, but the tiny added description of ‘white’ for the ruler indicates that he was familiar with the practical method. He gave these new constructions after largely following Gerle. They are simple perfect fifths, by divisions into 3 parts, from the Db, D, Eb on frets 1 to 3, giving Ab, A, B♭ on frets 8 to 10. There is nothing unusual here, since Ganassi and others had effectively done the same. The resulting tempered systems have been outlined above, and the frets could be adjusted. However, Gerle had already suggested a fret 8, which might have been preferable. The A on fret 9 is reasonable, but the B♭ on fret 10 would be very flat due to its derivation from the poor altered E♭ on fret 3. Gerle’s fret 3, or a perfect fourth from F on fret 5
would have been better. Even these straightforward additions of higher frets produce problems and tend to degrade Gerle’s elegant and economical scheme. A further practical point is that these 3 poorly tuned higher frets, or perhaps only 2, must be glued to the body, as Dowland himself wrote. These frets would certainly have needed later permanent adjustment, in addition to more flexible adjustments for other gut frets, but he did not mention this.

The main highly revealing ‘unwitting evidence’ occurs in his modifications for frets 3 and 4. He did not preface these changes, unlike his new frets 8 to 10, which indicates that he did not see or admit any great problem. His initial constructions were exactly the same as Gerle’s, and follow the same order. He apparently approved of the midpoint of the string, frets 7, 5, and their midpoint for fret 6, then fret 2, and even Gerle’s unusual fret 1. His instructions then clearly read as if he is in the actual practical process of constructing fret 3 using the division into 99 parts, and taking a midpoint with fret 5, when he sees that fret 4 is falling in a different position than he expects. It is important to note that Gerle wrote that fret 4 was just between 3 and 5, but Dowland took the more definite midpoint. The position he normally expected was surely 1/5th of a string length from the nut for a perfect major third or ‘ditone’ discussed above, as also set by Ganassi. For a midpoint, Gerle’s position would be 163/792 from the nut. We moderns can see by calculation that these ratios have values of 0.200 and 0.206, and the distances from the nut are 120 and 123.5mm for a 600mm string. The ancients could only have noticed this difference of 3.5mm by its position on the wooden ruler, relative to another mark for the previous familiar position at 120mm. This may have happened while writing for his son’s new book, in the past for his own use, or from someone else’s work, possibly setting up the master’s lute, etc. It is highly unlikely that the difference was predicted by inspection of the ratios, as in the above analysis. It is probable that Gerle did not calculate numerically a position such as 163/792 for fret 4, after the remarkable deduction of 16/99 for fret 3. Dowland then thought that he or Gerle has slipped up, and he wanted to retain his familiar fret 4. For this he needed to make a ‘neat little correction’ and move fret 4 closer to the nut, by about our 3mm. He noticed on the ruler that he could use the midpoint procedure if he moved fret 3 closer to the nut by our 6mm. This 6mm is just half of one part of Gerle’s final division between fret 1 and the nut into three parts, each of (2/99). Dowland’s fret 3 therefore produces a sounding length (1/99) longer than Gerle’s. Furthermore, we can see this is a ‘quick fix’ while trying out the construction, because he could have got exactly the same result by going one step back in Gerle’s method and taking just three of the initial eleven parts 3(1/33) to measure up from fret 1 to fret 3. Instead, he took the easier path of telling the reader to make a laborious halving of a ‘two thirds’ part, ½(2/99), and then subtract this from Gerle’s five ‘two thirds’ parts, 5(2/99). This gives 4½ (2/99) which he called ‘foure times and a halfe’, in order to arrive at three times one of Gerle’s initial eleven parts 3(1/33). There is no explanation or fuss over the change, just ‘four times and a half’ instead of ‘five spans’, plus the ‘perfect ditone’ explained above. It would have taken more thought and time for him to work out and write down the simpler method. Some extra work and explanation might have suggested he had keener interest. It would also have puzzled later explanation more than the messy quick fix, which provides us with good evidence. Another detail that indicates a quick fix is his estimation of half a part rather than a formal division into 2 parts, although he did later add frets 8 to 10 by formal divisions into 3 parts. The reason why Dowland wanted to take a strict midpoint for fret 4, which led to his alteration of Gerle’s original, is probably an asumption of the previous use of midpoints, or a wish to be definite.

There may be two types of overall explanation. The first is that Dowland might have taken up Gerle’s scheme near the time of writing, partly because it was in a precise easily copied form. He quickly worked through and needed to alter it, and then also added frets 8 to 10 as perfect fifths on 1 to 3. His ‘success’ in setting fret 4 as a perfect major third indicates that he was accustomed to starting with frets at perfect intervals. The very flat fret 3 that he had just formed would have been closer to fret 2 than fret 4. This may not have troubled him, since unequal...
spaces are common in meantones and initial set-ups, but not usually smaller between frets 2 and 3, and Ganassi’s frets 3 and 10 had been even worse. If his new fret 3 was not a surprise, this indicates he had no previous reference mark for this fret, or maybe it was Pythagorean between his new position and Gerle’s original. It is probable that less common perfect intervals, such as the minor third with a much shorter sounding length, or wide intervals, were not conventional simple starting points. He also left the rather sharp fret 1, which would not be ideal for a fourth course F♯ in his music, and he probably needed to flatten it later. A possible secondary ‘tastino’ fret is not mentioned by him or Gerle.

The second type of possibility is that his modified scheme had become customary for an initial set-up. It has been shown above that Dowland’s construction cannot be a final scheme and would have required considerable adjustment, but not so much as many others. It may seem surprising to us, but as seen from the present work, the ancients were not able to set up any initial scheme that did not need large later adjustments. Although the shift of fret 3 looks like a quick fix he may have seen no need to refine the method of construction for more long-term use. He may have been a little shocked by Gerle’s ‘elementary mistake’ on fret 4 and his ‘correction’ indicates that he had not fully understood Gerle’s purposes. As shown above, the difficult frets 1 and 3 were carefully constructed by Gerle to define a tempered scheme. It may seem a pity if Dowland savaged the marvellous fret 3, only to produce a distorted PYT system with an inconsistent perfect third. If this quick fix served his purpose then it is just part of the English empirical tradition, as opposed to theory for its own sake. His changing of fret 3 might indicate that he did not want the substantial tempering of an accurate SCM, and this would be in line with a later more relaxed use of Gerle’s scheme for a general tempering. However, he had moved his fret 4 in the opposite direction into QCM and this would require more adjustment towards a final less tempered system. Ironically, after these changes, his frets 3, 4, 5 and also 6, 7, 10, are almost exactly the same as Ganassi’s, which may have been in his mind during the work. He would have had to be content with adjusting most frets later, but he gave no instructions, unlike Ganassi. It is not easy to decide between the short or long term scenarios, but many of the above features are not exclusive.

Dowland’s final tuning would have needed to be a meantone, probably in the central region around ET and ECM, as outlined above. In light of the dependency on Gerle this conclusion may be loosened, because Dowland is now seen to be more acquiescent than inventive. Also the amount of scatter, and the ancient difficulty of accurately constructing frets for a specific tempering, mean that it is probably unsafe even to assume that his final meantone would be close to a simple average of his points on the diagram. However, his efforts to change Gerle’s fret 4 into a perfect third may be an indication of how he wanted to bias the final tuning for his music, even though this interval would not be truly compatible with anything short of QCM. The sharp thirds of ET may have been less acceptable than for modern ears. It is sometimes heard that fret 4 might be biased towards the nut, but most thirds on a lute involve other frets, and moving it more than a wishful nudge would cause problems elsewhere. On the other hand even SCM could give him a few difficult uses of enharmonic alternatives. These are explained fully in Ref 2, but their effects could be underplayed or exaggerated in performance. This is well known from uses of ‘musica ficta’ in vocal polyphony, where much larger differences such as say F and F♯ meet from different lines.

In some ways Dowland’s scheme just returns to the broad quandary between PYT and QCM, which we can reasonably narrow to ET and SCM. Whatever his preferences may have been for changing Gerle’s frets 3 and 4 it can be seen that the originals were superior for any degree of tempering. Dowland’s use of the standard claim of employing science rather than adjusting by ear begins to appear disingenuous, but it also saved him much more writing. We wish he had been less economical with his half page on fret positions. He seems in a hurry to round off with the thickness of frets relative to the strings. These details are worth listing, because it is clear technical evidence of interest today. Frets 1 and 2 should have the thickness of the countertenor,
or 4th course F; frets 3 and 4 like the great mean or 3rd course A; frets 5 and 6 like the small mean or 2nd course D; and frets 7 to 10 like the treble or 1st course G. The respective diameters would be about 0.8, 0.66, 0.53 and 0.4mm. It is possible that a use of well-twisted gut strings for the frets may have given more elasticity for adjustment than the tough gut usually supplied for modern frets. The large adjustments needed for final tunings may have required some new frets. Finally, he briefly tuned the strings with the pegs, from the bass upwards, with the lower string on fret 5, or 4. Then Robert presented some of his father’s finest pieces, written before 1610.

**Simpler and improved versions of Dowland**

Using the style of the ancient methods it is fun to see how Dowland’s scheme could be both simplified and improved. Frets 1, 3, 4 posed the greatest difficulties, which continue with frets 8, 10, 11. Fret 1 was set and then fret 3 was based on it, but the origin of these frets is Gerle’s construction for SCM. The main problem is seeking a perfect major third on fret 4 as the midpoint of fret 3 and the perfect fourth on fret 5. This just fixes a very flat fret 3. The numbers can be simplified with a perfect ratio of 5/4 for fret 4, leading to 20/17 for fret 3. This uses slightly smaller integers, but fret 3 clearly remains very flat. Perhaps surprisingly, these ratios are exactly the same as Ganassi’s scheme, but he also had a very poor fret 2.

Improvements can be found if a linear division is made between frets 2 and 5. This can allow fret 3 to be moved to another position. The simplest division is three equal parts, and after some calculation the frequency ratio for fret 3 is 108/91 or 1.1868, and the ratio for fret 4 is 54/43 or 1.2558. The great improvement for fret 3 places it between PYT and ET, and the value for fret 4 is close to SCM. This looks satisfactory for such a simple construction, and further complication is unnecessary. Fret 1 could also be simplified to the midpoint between the nut and fret 2. This gives a frequency ratio of 18/17, which is close to the ET semitone, and demonstrates the effect of a linear division of the 9/8 tone. Perfect fifths on these three new frets would lead to lower horizontals for frets 8, 10, 11, as explained above. The new scheme can be listed as:

1, 18/17, 9/8, 108/91, 54/43, 4/3, 24/17, 3/2, 27/17, 27/16, 162/91, 81/43, 2

The frets are shown as squares on the diagram, except where they are the same as Dowland’s circles. The modified scheme now has much lower scatter and lies between PYT and SCM, with an average close to ET. This takes account of all frets, whereas discussion of the original scheme discarded the rogue frets 3 and 10. This would be a good starting point for ET or ECM, and only moderate adjustment is necessary.

This fretting scheme is one of the simplest that could be devised from a Pythagorean basis with some equal linear divisions. The unusual steps of Gerle and Dowland are mirrored, but one cannot know how much they would have approved. The scheme could be called Modified Dowland, or perhaps Coakley’s fretting. This may double the number of schemes with an Anglicised Irish label. Most temperaments seem to have a German or Italian origin, with scarcely any from our isles, so we could do with some more.

In the same spirit, a better approximation to ET could start with the same Pythagorean frets 2, 5, 7, 10, and then fill in the other frets with 18/17 semitones. However, the normal repeated 18/17 rule could be set up quite accurately with less work. The construction on a strip can use a 1/18th division, then mark off one part sideways across the nut, draw a diagonal to the bridge, and read the successively smaller parts from the diagonal. In this rule five semitones give a frequency ratio of 1.331 for a fourth, then the fifth is 1.492, and the octave 1.986. The fourth is even flatter than perfect and beyond PYT, whereas the flat fifth is beyond TCM, which may be a shock for this popular method. The lower frets are close to ET but higher frets are more scattered and lead to the flat octave. The error in the fourth is similar to the Pythagorean approximation in the previous schemes, but the adjustment needs to be slightly greater. The error in the fifth is opposite but still only needs a small correction. Frets 5, 7, 12 would be flatter than exact ET by 1.3, 1.3, 2.1mm for a 600mm string.
The 18/17 rule is usually presented as a practical method in which there would be some uneven compensation of stretch-sharpening. For comparing different schemes, this is better left as a later correction, as in Ref 2. The above analysis shows that the 18/17 rule is not so exact as often supposed. The semitone of 1.0588 is significantly lower than 1.0595 for $2^{1/12}$. A perfect fourth contains two 18/17 semitones, two of the larger type $17/16 = 1.0625$, and the small Pythagorean diatonic semitone $256/243 = 1.0535$. The rule needs a type of adjustment different from the other schemes, since there are no perfect reference points, and not even the octave. It is possible that the ancients, such as Galilei, arrived at this rule by first testing five semitones against a perfect fourth, and this composition of a fourth was well known to the early Greeks. It seems natural to think in terms of ‘filling in’ the fourth, and precisely the same verb is used in the Timaeus, as discussed above. The differing approaches of filling in intervals, building up scales from tones, or generating notes from intervals are worth some general thought.

In the list of fractions I derived above, $160/151 = 1.0596$ is seen to be a highly accurate alternative to 18/17, and maybe previously unknown. The simple factors of the numerator require five successive 2 part divisions to produce two 18.75mm parts near the nut, then a 1/5th division on just the second part from nut, and finally marking a 3.75mm part back. This places fret 1 at 33.75mm from the nut. This would be no harder than the 18/17 method, and also gives an exact ET to compare with other schemes. The analysis also shows the degree of accuracy needed for placing successive frets, which are prone to error even though the initial construction looks simple. This is similar to the discussion of Pythagorean constructions. The 18/17 rule needs a 33.33mm mark across the nut, which is about ½ mm less than for a 160/151 method. This sort of precision is achievable and a neat practical approach might involve repeating a construction in order to produce a final check of a good octave. Among the previous fractions that I derived for ET, $125/118 = 1.0593$ requires three 5 part divisions, similar to the ECM scheme. Further fractions are $71/67$, a very accurate $89/84$, and $107/101$, but these have difficult numerators for making divisions.

These simpler schemes raise an obvious question obscured by the elaborate theories. If the ancients had earlier made final adjustments towards ET, with its obvious uniform geometric progression of fret spacings, it would have been very simple and quite accurate just to tell the reader to ‘fill in frets 1, 3, 4, 6, 8, 9, 11 so that the spacings reduce evenly, then slightly adjust frets 2, 5, 7, 10’. Many probable factors have been discussed above. During the modern lute revival, fret positions have doubtlessly been taken from the nearest guitar to hand, with good final results.

The new ratios found above for an ECM version of Gerle’s scheme can be expanded into a complete set of frets as:

1, 120/113, 9/8, 25/21, 200/159, 4/3, 24/17, 3/2, 180/113, 27/16, 25/14, 100/53, 2

Frets 1, 2, 3, 5, 6, 7, 9 have been discussed above. Fret 4 is again the midpoint of 3 and 5, but now falls exactly in the same ECM system, as shown on the diagram by the solid diamonds. This was not possible for Gerle, who would surely have liked this feature. The scheme could also be precisely what Dowland wanted, but he might have tried to move fret 4 down to QCM! For this he would surely have known the construction for a perfect third of 5/4 by a 5 part division, and this is also the first step for the ECM. For interest only, a perfect third would be a small ‘Ptolemaic’ semitone of 21/20 above fret 3. Frets 8, 10, 11 are simple perfect fifths on frets 1, 3, 4, and hence moved to systems with slightly different tempering, as explained above. It is easier to see that fret 8 would be at 51/32. The fret positions have much lower scatter than all the above schemes, and mainly in the slowly varying Pythagorean approximations. This scheme might be called Modified Gerle, Dowland’s Dream?, or Coakley2.
FINAL REMARKS

A recapitulation may be useful in view of the length and number of topics. Some new matters such as modern choices of tuning and the ancient modes also need attention.

This paper started as a short assessment of the usefulness of Dowland’s fretting instructions for setting a practical tuning. A new method used the diagram from Ref 2 showing all possible meantone temperaments. This displays at a glance how close all the notes would be to a single consistent tempered scale, as represented by a horizontal. This is much clearer than trying to compare several intervals in long lists of decimal points and cents. It may still be interesting to set up frets from old schemes and attempt to tune and play. The scatter of Dowland’s notes over various temperaments showed that considerable final adjustment would have been necessary. It was concluded that his final tuning would have needed to be a meantone, most likely in the central region around ET and 1/8th comma.

The focus then moved to an investigation of Gerle’s earlier fretting scheme, which had been identified as Dowland’s model. Analysis of Gerle’s constructions confirmed a description of 1/6th comma meantone. The new work showed how Gerle used a variety of less common ratios for the semitone, which in turn led to ingenious geometric constructions for his fret positions. The reasons for the exact form of his chosen ratios could be understood for the first time. These ratios show in detail how Gerle and other ancients thought about tuning. This is more enlightening than an immediate mechanical conversion of fret positions to decimal points and cents. The new analyses depend on thinking more generally how the ancients used geometry and arithmetic to provide instructions and measurements for making things. Investigating how different degrees of tempering might be constructed led to the surprising conclusion that there may have been very few if any possible alternatives for Gerle, without resorting to much greater prohibitive complication. During this search a single new scheme was found by a method slightly different from Gerle’s. This has accurate 1/8th comma tempering, which could have been close to what some ancients wanted, but was most likely unknown to them. It was concluded that Gerle’s scheme probably came to be used later for adjustment towards equal temperament, and maybe even 1/4 comma meantone. This new work also questions how precisely the degree of temper was specified or desired by theorists and composers.

Some further details in Gerle’s work led to a new analysis of the early methods for Pythagorean schemes. This showed that the constructions were lengthy, prone to error, and that it was not easy to set certain frets in the early schemes. It was seen that Gerle’s shorter method dealt well with the difficult frets that define tempering, and also had features that reduced errors. It may have evolved as a way of solving practical problems with Pythagorean constructions. Ganassi’s simple geometric placing of frets to give some perfect intervals was also compared with the other schemes.

Reviewing Dowland’s instructions in the light of Gerle, the initial conclusion still holds but less strongly because the scheme was evidently borrowed. Some effort was spent trying to explain in detail why Gerle’s particular scheme was used, and why, how and when Dowland made several strange modifications. For the most puzzling change it was concluded that he wanted Gerle’s fret 4 to give a perfect major third. This would have been the expectation of the simple classical tuning theory he described in the preface to his scheme. Dowland probably sought good thirds for his music, but in a final tuning this note could only belong to the unlikely highly tempered 1/4 comma system. To follow Gerle’s method Dowland thought he had to move fret 3 to be extremely flat in an oppositely tempered system, way beyond Pythagorean. However, he could simply have placed fret 4 independently and left fret 3 unchanged. His frets 3 and 4 became far apart on the diagram and would need much later adjustment. In a final judgement, it seems unnecessary work just to satisfy the expectation of simple theory, particularly when this would require greater later adjustment. Nevertheless, in support of Dowland’s fretting, the scatter and
necessary adjustment are still less than several other ancient schemes that have received greater modern regard, and he added two higher frets.

The factors involved in his choice of Gerle’s method are its reputation, concise form, practical instructions, no lengthy advice on adjustments, frets which would require less adjustment than some other schemes; ease of following, copying and altering; its availability and a continuous use; and the chance of being at hand when writing for his son. A major reason why a 70 year old tuning was still relevant for a later musical style could be its continued usefulness for setting up any type of meantone. This can remove at a stroke the difficult puzzle of why Dowland has seemed to employ an outdated scheme with stronger tempering than might nowadays be thought appropriate for his music. It is possible that he obtained it early in his career nearer to its publication, perhaps from a teacher, and then grew up with it. His later writing about Gerle’s book is in fact important primary evidence that these theoretical fretting instructions were of interest to other musicians. It is sometimes assumed, without evidence, that nobody paid much attention to the theorists. Similarly, it may be seen that Gerle responded to difficulties with the earlier Pythagorean methods. All these aspects are finally just as important for us as technical details of the changes and mistakes in Dowland’s writing.

A key result of the present work has been an understanding of the ancient fret constructions. It has been shown why they indicate final tunings less precisely than might have been expected by modern readers. Sometimes deeper explanations reveal uncertainty, and previously firm conclusions are seen as assumptions. It is also worth stating clearly that old lute fretting recipes which may appear to be different from some type of meantone are not usable special alternatives, but just a result of the ancients coping with their available understanding and methods. Since this new understanding does not tell us the precise final tunings used by the ancients, these may be better suggested by the music.

Many fine modern players try to use a temperament suited to a particular piece, type of music, composer, era, etc. People who have studied the historical, musical, and scientific aspects would fully endorse this. Most writing on old tunings has concerned the musical use, but it is very laborious to analyse the effect of various temperaments, even if some common examples are chosen from whole pieces. Playing through may reveal all the differences more easily and fully. Some players may use ET for convenience, but still acknowledge other temperaments. After the present work on Gerle et al, a more relaxed view might be taken of the appropriate uses for temperings of 1/6th, 1/8th, 1/11th comma, etc. The main purpose of the present work has been understanding, but this does not alter the range of tunings between SCM and ET that has been found useful for 16th century music. This may be viewed as a positive support rather than a lack of novelty.

It might, however, be interesting to experiment between SCM and QCM for even richer consonance in some pieces. One of the main purposes of ET is to spread tuning problems over all keys. In contrast, with stronger temperings the defects are pushed into unused keys that were unnecessary or inconvenient for lute technique. For someone who has been satisfied with ET, it may be interesting to know that the main effect noticed by a lute player moving back to ET, from the nicer keys in say SCM, is a feeling of confused brilliance. If SCM seems too large a change from ET, but ECM not too different, then one could try 1/7th comma. Similarly, 1/5th comma might give easier fret spacings than QCM. Earlier music could explore between PYT and ET, maybe with 1/15th comma. The original players using the ancient schemes probably strayed into these areas. All these fret adjustments can be made from the diagram, where it is accurate to use simple linear interpolations between the five main examples and even between PYT and TCM. In terms of the low accuracy of the ancient methods, it even becomes significant to state that all the schemes examined above, and many others, do in fact fall within this large region, apart from some odd exceptions by Ganassi and Dowland.
Modes were used in 16th century lute music, such as the fantasias etc that were imitations and arrangements of vocal polyphony, and held to be the highest form. For the most famous master, da Milano, most of his surviving pieces are in this style, but there does not appear to have been a specific musical discussion on appropriate tunings. Since the vocal ideal would employ flexible natural tuning it might be thought that these modes required many special lute tunings. Initially, this seems a forbidding problem, even with the constraint that many intervals must remain well tuned. After some thought, the problem can be largely removed, by reversing it and noting that meantones are the only possible consistent tuning on a lute. Then for better consonance of common intervals in order to approach vocal tuning, greater tempering towards SCM and even QCM may be appropriate, but perhaps with different enharmonic choices for some frets. This could be more satisfying than a simple default position of ET, which seems to have been used on the few, otherwise very fine, recordings of da Milano. Written evidence shows that the original vocal masters such as Palestrina used a lute for demonstrating their music, and possibly as an aid to composition, similar to the use of the clavichord and later the piano. Any use of tablature would superimpose the parts, and this might have been an additional aid to the normal ancient use of several separate staves. There may also be useful relations with similar keyboard masters such as Cabezon, where a greater choice of tunings would have been possible.

The inadequacy of ‘iconographic’ evidence for geometric measurement has been discussed extensively by others. For example the proportions of people are often highly distorted, particularly when grouped to make a pleasing composition. Most paintings I have seen from this early period, up to about the mid 17th century, show equally spaced frets, which is not a surprising approximation. If one assumes correct perspective had been used, some cases would even imply fret spacings that decreased towards the nut. The greatest masters are ‘equally at fault’ here. While Caravaggio may not show clear spacings of actual frets, his glistening fret knots are equally spaced, and perhaps his musicians and other youths should warn us off assuming photographic accuracy. Other easily accessible evidence has appeared on the front cover of many recent editions of Lute News. Equal spaces might be more understandable for the rather confusing alternate wide and narrow spaces of say SCM, than for the more striking simple gradation of ET. Some disarray is seen when a player’s hand overlaps the frets, which will be understood by anyone who draws. Later paintings of baroque lutes, including types with long necks, do indeed show fret spacings characteristic of ET, and it is generally assumed that this became fairly standard. This now merits theoretical examination, because initial thoughts suggest that tuning constraints between courses may differ from the renaissance lute. It could be tempting to view Dowland as an intermediate case, but this may not be a perfect guide to his final tuning. One of the last notable references to temperaments is in J S Bach’s keyboard works, but even here there are no unique solutions, so any difficulty in studying much earlier practice should not be too surprising.

Ancient education centred on the quadrivium of three sciences together with music. We can see that Dowland’s writing was actually quite scientific, while he let his music speak for itself. Modern musicians, especially in these isles, have been subjected to the ‘Two Cultures’ split into sciences and arts, and generally end up in the latter camp, deprived of the technical background enjoyed by Dowland. (With degrees and research in both areas, I could make further balanced comments. It is also notable that both Barbour and Lindley came from North America.) In recent times tunings and temperaments have been viewed as a small but difficult complication, separate from musical theory. This may be fine for the almost universal modern use of ET, but not very helpful in the revival of early music. Ref 2 tried to assist in explaining all the steps in setting a meantone, and provided equations, tables and the diagram of fret positions.

The modern player of early music has two main practical tuning interests: frets and meters. Generally, fret positions have been compiled from modern theory, and then used by makers for their customers. This may be sufficient for many players, but some may have been tempted by
the invitation to ‘enter’ a string length for unusable schemes, as on the LSA website. Such a filter might explain why nobody announces that they are using, for example, Dowland’s or Ganassi’s temperament. By chance, tuning meters became available at the start of the recent early music revival. They are particularly suited to tuning ET, but extra theory is needed for alternative early temperaments. Even the most expensive meters have only a few options, which may severely limit a player’s horizons, unless they learn some theory. This practical restriction could establish some peculiar ideas about ancient tunings. Ref 2 gave a small three-line table (p15) showing how deviations in cents for any type of meantone could be read on an ET meter, for setting frets and tuning open strings. A following paper gives new details using the Korg C40 meter.

For a diversion one could deduce or confirm a performer’s tuning with the aid of a digital photo, a short ruler and a calculator. Only two fret spacings are needed, and those with the greatest potential difference such as 1 and 4 would be best. An inverse solution for the fourth is difficult, so a simple list of values from Ref 2 would be sufficient. The spacing between frets 1 and 4, divided by the spacing of fret 1 from the nut, has the range of values: 3.1 for PYT, 2.7 ET, 2.4 SCM, 2.1 QCM, 1.8 TCM. A lute spotter is likely to encounter a value between 2.8 and 2.2, but some test runs would be helpful.

Finally, the modern disregard of Dowland’s instructions has probably arisen from expectations of a master giving us his personal fine-tuning, and then not even finding one with sufficient modern accuracy for general use. The result is similar to that of many great creative figures later in life volunteering to explain their work. The small enticing book titled ‘The Meaning of Relativity’ by Einstein himself is also a little disappointing.

Nevertheless, this study has drawn out some interesting new technical and historical conclusions. My feeling for Dowland’s writing has changed from curiosity, to frustration, annoyance, and finally some satisfaction, mainly from a new understanding of the ancient methods and Gerle’s impressive scheme. Dowland’s famous but erratic scheme has directed attention to many other areas that needed explanation.

A major aspect of the present work has been using science and close reading of sources to understand the ancient methods. This can be much more fruitful than an approach conditioned by familiarity with modern technical precision and engineering, which can throw up misleading questions and conclusions. A previous enquiry into tapered gut strings benefited from the new approach, and some further work on this topic forms a short paper. The study of fretting schemes has also been extended by work on the actual fixed frets on surviving citterns and orpharions. This forms another very large paper.

During this study I have recalled Ian Harwood enthusiastically pointing out to me Dowland’s handwritten tablature in the Folger manuscript, in the fine Painted Hall at Greenwich. For his many helpful lute-making articles he was content to end the long process by setting ET frets, but referred with interest to other schemes. Also, this paper coincides with the happy 450th anniversary of Dowland’s birth.

References
2. C. J. Coakley: FoMRHI Comm 1808.
Key to diagram
Dowland’s fret positions are given as circles.
Gerle’s fret positions are shown by open diamonds, and by circles where the same as Dowland.
Ganassi’s fret positions are shown by crosses, and by circles where the same as Dowland.
Coakley’s fret positions are shown by squares, and by circles where the same as Dowland.
Coakley2 fret positions are shown by solid diamonds, and by circles where same as Dowland.

The diagram contains an enormous amount of information, even without the practical examples. The five schemes can be compared by selecting a symbol and following it across the frets with a fingertip. For greater clarity one could make several enlarged A4 copies and highlight with coloured pens. (The angled lines are not slanted frets.)