FoMRHI Comm 2165 **Adrian Geisow**

Fingerboard geometry - further comments on Comms 2143 & 2153

David van Edwards[1] has suggested that I comment on the mathematics of fingerboard geometry discussed in 2143 & 2153; in particular the derivation of an 'ideal' shape of scoop or relief along the line of a string, to allow for low action when playing.

First though we should discuss a common misconception about fingerboard shapes. Given that the fingerboard on a string instrument should presumably lie as close to the strings as is practical, to minimise the effort in playing, we first consider the shape of the (virtual) surface in which the strings themselves lie and derive the optimum fingerboard shape from that.

A confusion of cones…

In 2143, Munck[2], along with many others[a], makes a common assumption that if the string ends at the bridge and the nut both lie on circular arcs, then the surface that joins them is conical[b] (or cylindrical if both radii are equal), and that therefore this should be the approximate shape of the fingerboard. Munk then points out that the strings may not align with this presumed cone and from this explains a 'scoop' in the fingerboard, presumably observed in the 1619 Jaye he discusses. The misconception is well described by Jaen[3], but I will summarise the key points here.

The actual surface we want is called a *ruled surface*[c], where each point on one curve is joined to a corresponding point on the other by a straight line – the strings on an instrument following a handful of these lines. To see the whole surface, add more and more strings in between the usual ones to gradually fill in the spaces.

An equivalent approach that explains the name would be to lay a straight edge on both curves, and then smoothly drag it over both – the surface traced out by the straight edge is a ruled surface. In principle one could start with the straight edge joining any two arbitrary points on the curves and move each end at an arbitrary and perhaps varying speed, so there are an infinite number of different ruled surfaces passing through the two curves.

Cones and cylinders are but a small subset of the set of all ruled surfaces – and we can choose *any* convenient curve for the bridge, and any other for the nut, independently, and then choose how to connect them, and we end up with a coherent surface in which all the strings lie. The key point is that although such a surface may appear 'scooped' from some angles, *it is perfectly straight along the strings,* by definition – remember here we are

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[a] an online search for 'conical' or 'compound radius' and 'fingerboard' or 'fretboard' generates a lot of results.
[b] Note that 'cone' is often taken to mean a very specific 'right circular cone' – to quote Munk "*imagine [a…] parking cone*". For a right circular cone, the circular arcs lie in a plane orthogonal to the cone's *axis* – whereas circular sections measured along a fingerboard surface will be orthogonal to the surface, rather than to the imagined cone's axis, and thus at an angle to said axis - so any such cone would have an elliptical rather than circular section.
[c] Jaen appears not to have known of the term 'ruled surface', but the CAD system he used implemented them.

describing the surface in which the strings lie. The same can be said for the fingerboard – if I draw a curve offset from that of the nut, to correspond to the desired fretting height, and another offset from the bridge curve, and then drag my straight edge over these curves, I'll trace out another ruled surface which could be the fingerboard. Munck points this out: *"it is easy to check the geometry of the fingerboard by holding a straight-edge precisely along the trajectory of each of the strings"*

We can see that any 'scoop' that is observed in the fingerboard, when checked by holding a straight edge in line with the strings, is nothing to do with the geometry implied by cones, upside down or not. An exaggerated example of the geometry described by Munck, with a smaller radius bridge than at the nut[d], is shown in figure 1. We can see that the straight 'string' lines do not intersect at a common point, so the surface, though well defined, is not any kind of cone; nor is there any 'scoop' along the strings.

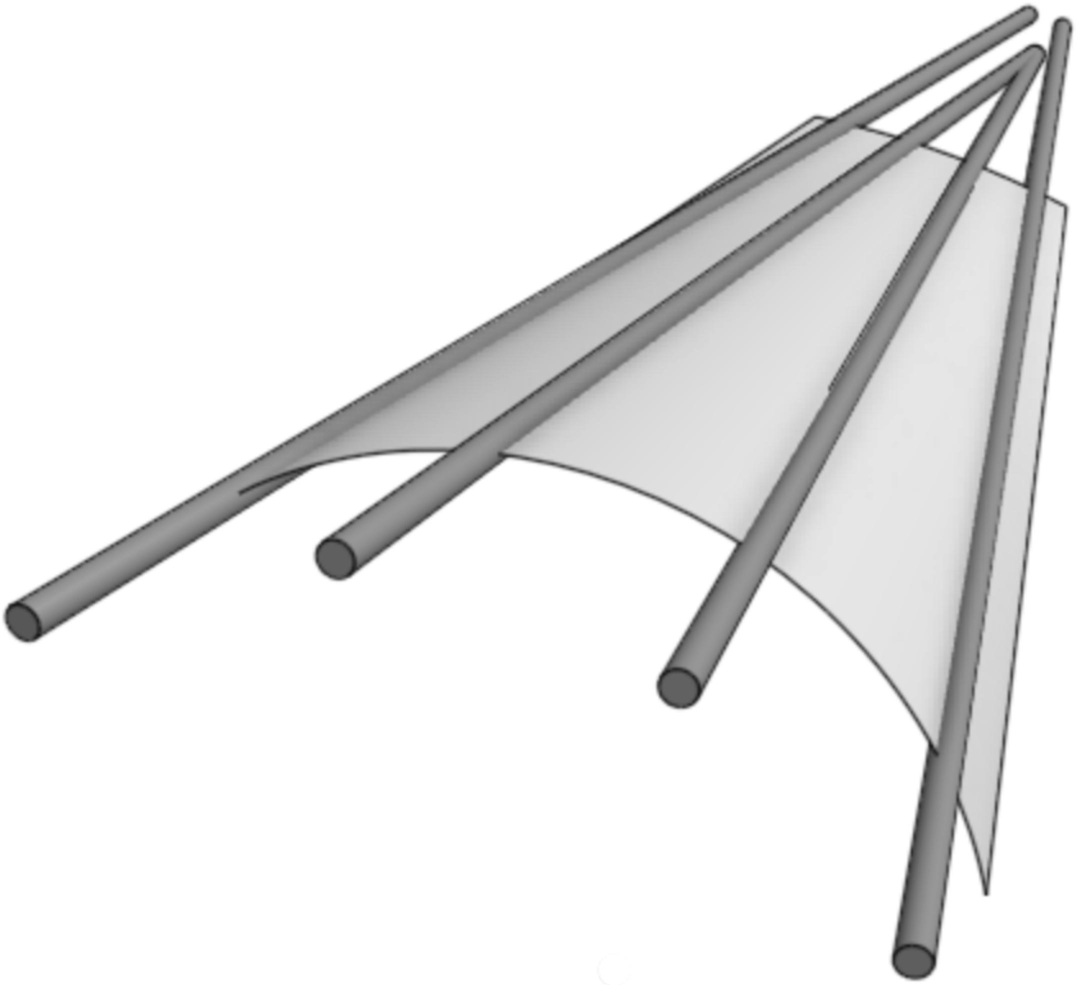

Figure 1

[d] This geometry is used on some modern instruments, for example the electric bowed instruments from NS Design[11].

Whatever the desired curves at bridge and nut, a ruled surface is an entirely natural consequence of a straightforward way of making a fingerboard: after any rough shaping, the maker planes, or sands, along each string trajectory, giving a straight facet under each string, then blends these together. Fingerboards can even be twisted, with claimed ergonomic benefits[4,5].

Optimising the fingerboard

Now for instruments with significant amplitude of string vibration – particularly long, bass strings on instruments such as viols, cellos etc. – we will find that a ruled surface is slightly sub-optimal. To have sufficient clearance when stopping the string near the nut, we will have too much clearance near the bridge end. This is where a deliberate 'scoop' is desirable, to minimise that action further up the fingerboard whilst retaining sufficient clearance to enable the string to vibrate freely. As far as I can tell, this has mostly been implemented by makers' rules of thumb – for example Kwan[6] surveyed several violin makers for measurements of their instruments, but to no great conclusion.

In ~2008, David van Edwards asked me if there was an ideal shape for this scoop. A simplified presentation of this follows, considering the shape along the path of an individual string.

Problem statement

Given a string stretched between nut and bridge anchoring points, what is the optimum shape of the fingerboard curve (*f* in figure 2) such that the string always just clears the fingerboard when vibrating, wherever it is fretted?

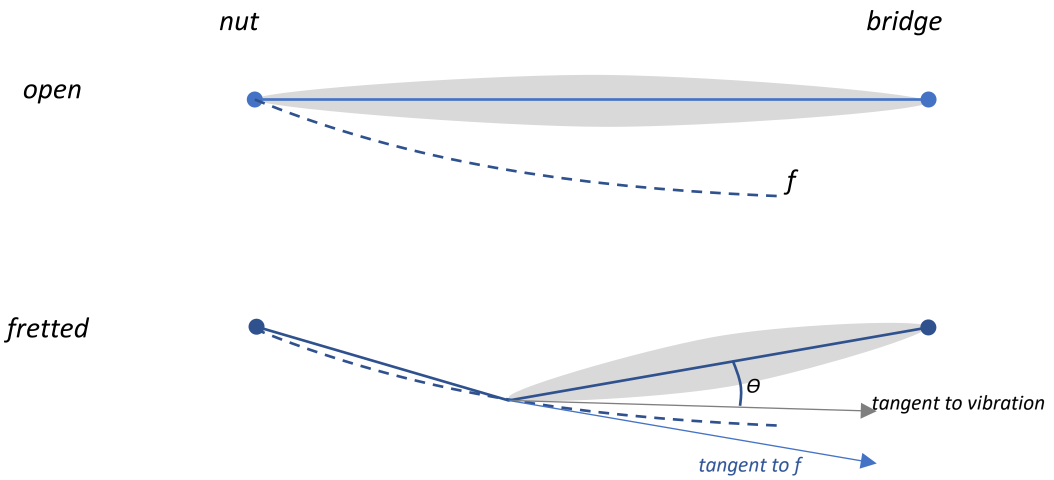

Figure 2

Assumptions

We assume an 'ideal' string – infinitely thin and flexible, and that the pluck force is set at some fixed maximum for practical purposes. The shaded areas represent the maximum extent of string motion. For fretted instruments, the curve *f* represents the line through the top of the fret surfaces.

Analysis

It is clear from figure 2 that the fingerboard curve *f* must always clear the shaded grey vibration extents to avoid buzzing. The optimal curve is therefore the total envelope of each of these extents considered together, as the fretting point moves along that curve. This means that the tangent[e] to the fingerboard curve at the fretting point should be aligned with the tangent to the lower portion of the vibration extent at that point. As illustrated above, the curve *f* could be made shallower without impinging on the string vibration; if the tangent to *f* were to be above the vibration tangent, the string would hit the curve.

So we need to find an expression for the vibration tangent at each point on the curve, as this will determine the shape we require.

Extent of vibrating string

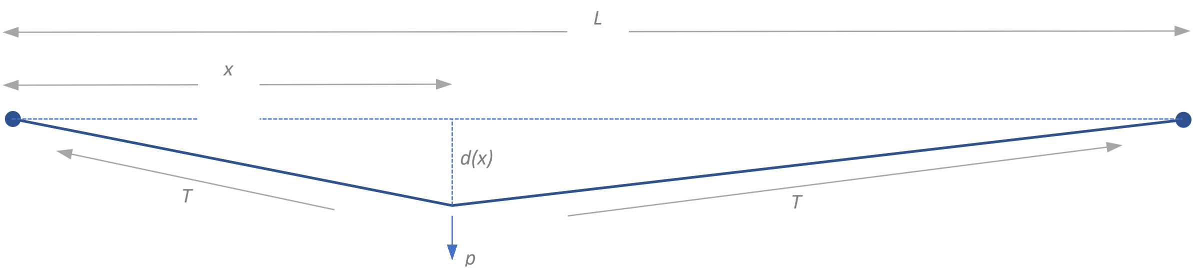

Figure 3

Consider the string above of length *L*, at tension *T*, and plucked (or bowed[f]) at a distance *x* with a force *p*. The deflection *d(x)* will be given by balancing *p* against the tension[g] in the string, i.e. when

$$p = T\frac{d(x)}{x} + T\frac{d(x)}{L-x}$$

We can rearrange this to give

$$d(x) = \frac{p(L-x)x}{LT} \quad (1)$$

which is a parabola with a maximum deflection of $\frac{pL}{4T}$

No matter how the string vibrates once plucked, the maximum possible deflection at *any* point *x* along the string is given by the equation *d(x)*, independent of where we initially plucked, as to do so would require more energy than has been put in by that pluck.

Using approximate values for an example classical guitar string, T=62N, L=0.6m, p=1N (about 100g), we see the maximum deflection is about 2.4mm (figure 4). For most playing styles, most of this displacement is from side to side, rather than up and down; here we need only consider the component of the displacement towards the fingerboard – this will typically be

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[e] ie the slope of the curve.

[f] For this analysis, we can regard bowing as repeated plucking as the string sticks and then slips against the bow.

[g] We ignore the very slight increase in tension caused by the string's deflection.

a rather smaller value. After release, the kink in the string initially at the pluck point bounces back and forth *along* the string, gradually reducing in amplitude (until the next pluck or stick/slip). This motion is not intuitive but has been shown by high-speed filming[7,8].

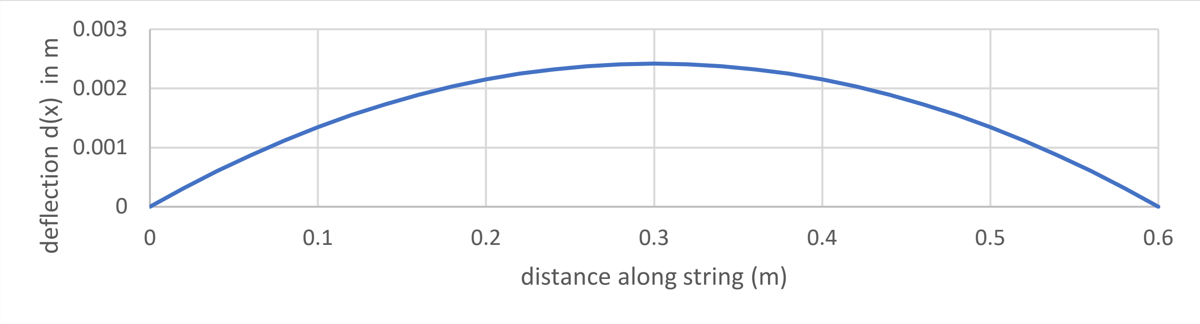

Figure 4

As it is the tangent that we need, this is given by the derivative of equation (1):

$$d'(x) = \frac{p(L-2x)}{LT}$$

Which at *x=0* is

$$d'(0) = \frac{pL}{LT} = \frac{p}{T}$$

We have the rather nice result that the derivative at each end is just the ratio of the plucking force to the tension in the string, *regardless of the length of the string*. This means that the angle ϴ on the first diagram has the same value, wherever we fret, and is just a function of this ratio. From now on we'll call this ratio *R*, as for the purposes of this analysis we don't need to vary both *p* and *T*.

In the original version of this paper, I made rather heavyweight use of some mathematics software to derive the resulting fingerboard shape, which can be expressed in parametric form as follows:

$$x(t) = -e^{-t}L\cos(R\,(-t)); \qquad y(t) = e^{-t}L\sin(R\,(-t))$$

where we use *-t* as the parameter as that conveniently gives us the nut position at *x* = -*L* at *t*=0 and the curve spirals towards the bridge at *x = 0.* The exaggerated curve is shown in figure 5 for *L* = 1m, and *R* set to 2 (implying a plucking force of twice the string tension!).

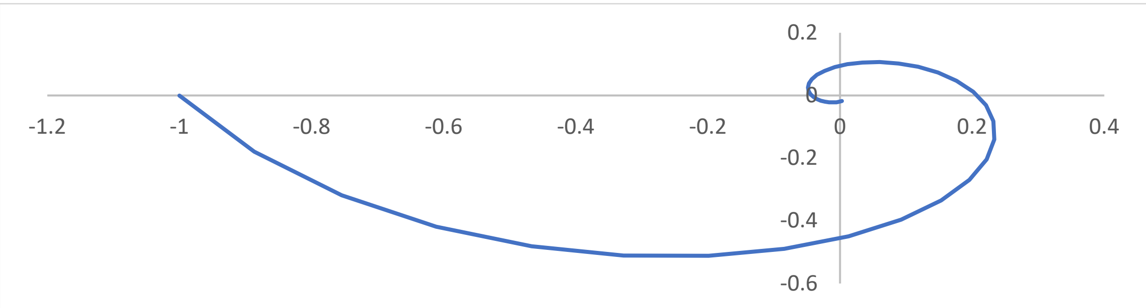

Figure 5

This curve is known variously as the *logarithmic spiral, spira mirabilis* or *equiangular spiral,* and is commonly seen in the spiral growth of shells. It is this last name that gives us the clue to a more straightforward derivation – it is called *equiangular* as the line drawn from the origin to any point on the curve always makes the same angle with the curve at that point. We could have simply observed that the requirement for a constant value for the tangent to the vibration extent directly implies this spiral. A more realistic example plot is given in figure 6 for an R value of 0.02, i.e. a pluck force of 2% of the string tension. In this case the maximum distance from the open string line to the fingerboard surface is about 4.4mm.

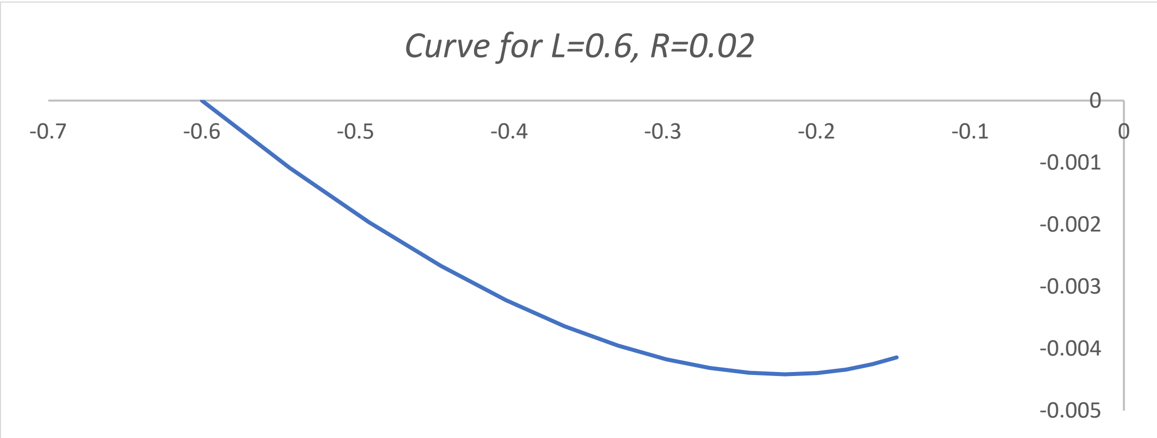

Figure 6

Notes

The curvature steadily increases towards the bridge. I've only plotted up to 15cm from the bridge, as that would correspond to the 24th fret position; the *y* scale is exaggerated. The curve is fairly flat before the 18/19th fret position – a straight edge laid between the nut and the 19th fret position would show a maximum deviation of ~1mm for this plucking force/tension ratio. Of most interest is that the scoop is concentrated at the bridge end of the fingerboard. For instruments such as guitars, the slight bend in the neck induced by the string tension usually gives plenty of curve – modern steel strung guitars use a truss rod which can be tightened to reduce this bend.

Caveats

This was for an ideal string; real strings are stiff and have appreciable thickness. Stiffness will restrict the range of motion relative to the ideal case, so this analysis should be on the safe side. To account for string thickness, we should offset this curve by half the string diameter, but this will have a negligible effect for normal string thicknesses.

Prior work

The use of the logarithmic spiral has been proposed before, but I've not been able to find a coherent presentation in the literature. The closest is Liu Jingye[9], who assumes that a constant angle between stopped string and fingerboard is desirable (versus a 'straight' fingerboard, where the angle will increase as one moves up the board) but does not justify the *why* a constant angle is desirable. Interestingly Segerman[10] in FoMRHI 71 compares straight, parabolic and logarithmic spiral shapes, again stating that the latter gives a constant angle between string and fingerboard, but again does not justify *why* that is appropriate. He also refers to 'inharmonicity and energy extraction expanding the vibration envelope' – it seems to me that such factors would if anything limit the envelope, but I may be missing something here.

References

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