Fomhri Comm 2164 **Chris Egerton**

Equal temperament fret position determination using proportional dividers[1]

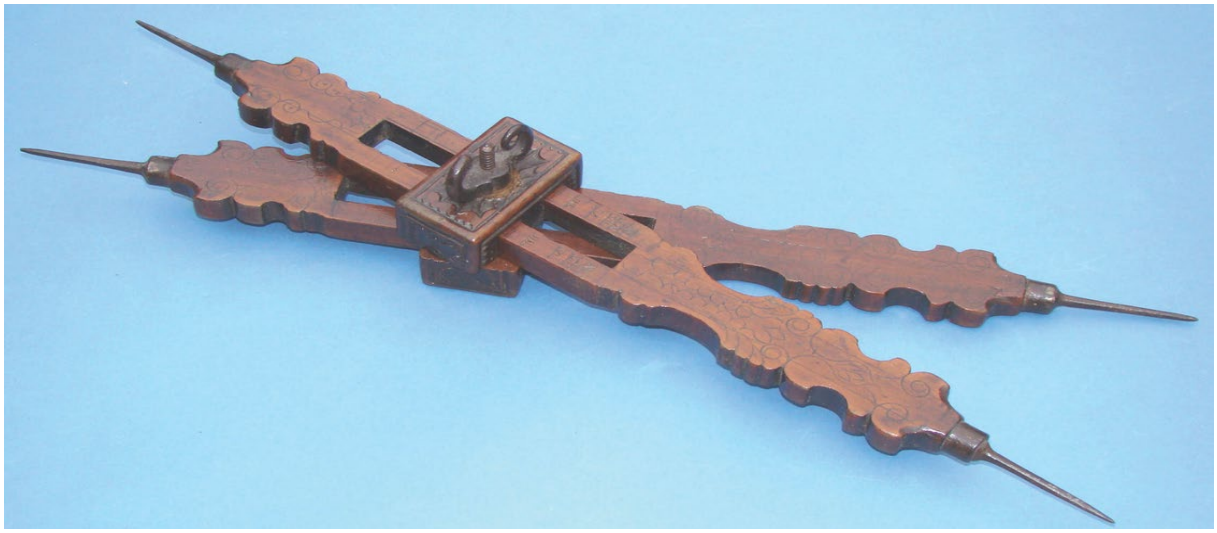

Plate 1: Proportional dividers, carved wood with iron points. Inked and engraved division scale and scrolled decoration. Probably central European, c. 17th / 18th century. 39 cm long.
(Image courtesy of Tesseract.com).

Abstract

Proportional dividers[2], the predecessor of the Sector[3], were known and used in the 16th century and later, for determining and calculating proportional relationships of length, area, volume, weight, as well as facilitating geometrical work with polygons, circles etc., all without complicated mathematics. Leonardo sketched several proportional dividers in his notebooks[4]. As with the Sector, the dividers rely on the Euclidian proposal that similar triangles are in proportional relationship[5]. The dividers can easily be used for fret position determination based on the 'Rule of 18'[6]. I describe a simple experiment and method that requires no maths, geometric drawing or particular skill.

Introduction

Proportional dividers were used mainly as a drawing instrument and working tool by map-readers, navigators, artists, sculptors, educated professionals etc. Historically, proportional dividers were hand-made to order for specific needs. The manufacturing precision and accuracy of the engraved calibrated scales depended on the skill of the mathematical instrument maker[7]. Boxwood, ivory, brass, silver, nickel alloys and steel were typical materials used due to their inherent durability and wear resistance. The tool resembles a double-ended dividers secured by a central pivot that is adjustable and permits variable setting to the calibrated scales inscribed on the legs. (**Plates 1** and **3**). The scale types vary according to the maker and the intended use.

History and use

If any map-maker, architect, artist or sculptor wished to proportionally scale up or down (in size) a drawing or an object, or a musical instrument designer wanted to draw and make working templates for a matched 'chest of viols' or another size-graduated set of instruments, proportional dividers were an essential tool for efficient working. Their design and function however, has some limitations that were solved by the later refinements of the 'newly invented' Sector[8]. Proportional dividers are very effective for simple fractional conversions such as 4:1, 2:1, 5:1, 4:7, or for equal division of odd distances or lengths, but much more complex mathematical and geometrical calculations are possible, if the operator is educated in those disciplines and the relevant scale calibrations are inscribed. They were still in common professional use in the late 19th century and are still used today as a helpful drawing instrument, or functional tool, for example by some organ builders and restorers. Accuracy of results depends on careful transfer of measurements and the stability and accuracy of the instrument's setting. Modern versions of proportional dividers are often very precisely engineered; the best achieve accuracies of tiny fractions of a millimetre on a typical dimension conversion. Some are equipped with micrometer-style adjustment that allows very high precision.

Many surviving historical proportional dividers are typically 9"- 12" in size, but much larger dividers are known (**Plate 2**) and they are ideal for direct calculation of fretting for long scale-lengths.

Plate 2: Large proportional dividers in brass and steel. French, late 17th early 18th C. 67 cm long. (Image from Le Zograscope on fleaglass.com)

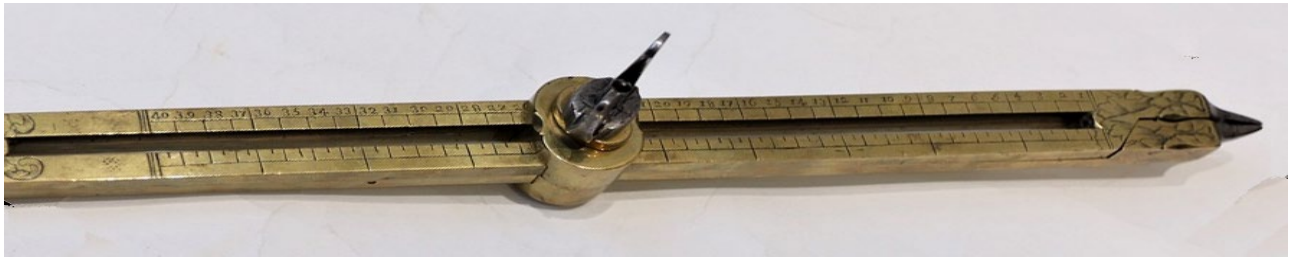

Plate 3: Detail of scale markings on dividers in Plate 2 (image ex: Le Zograscope)

Apparatus
For this article I made a simple experimental proportional dividers from 5mm x 25mm wood about 650mm long, with 1mm diameter veneer pins for points and with a bolt setting for the proportion of 18:1 at the long and short divider points respectively (**Plate 4**). A setting of 18 inches at the long-points therefore automatically reads 1 inch at the short-points. I used a scale length of 600 mm for my experiment, but it was not necessary to know the units or measured value as the dividers do all the work.

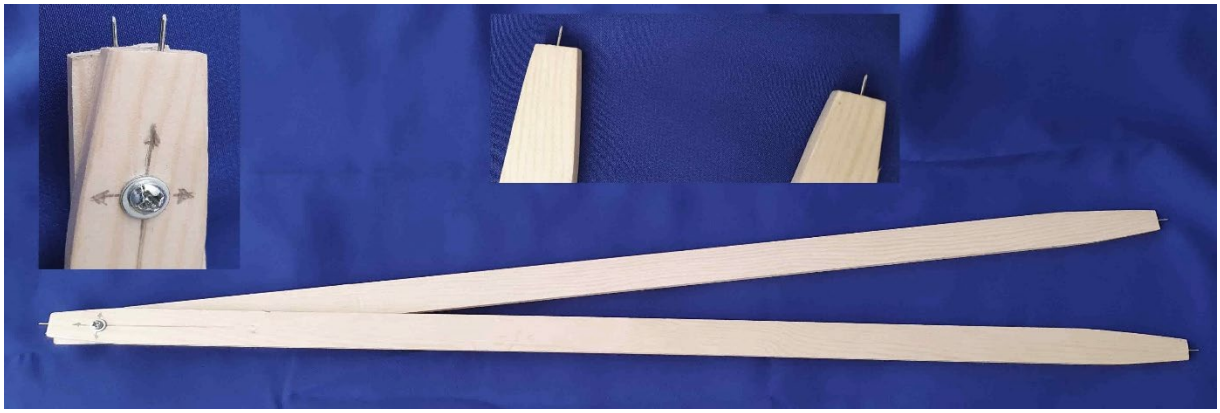

Plate 4: User-made proportional dividers with 18:1 setting.

Method
(Presented in instructional form for convenience)
1 Draw a straight line (e.g. on your working drawing) and mark off your total desired scale-length, noting one end as the 'nut' and the other as the 'bridge' datum point.
2 Open the dividers and set the long-points to the full scale-length dimension.
3 Without disturbing the setting, uplift the dividers and from the nut, mark off the *first* fret of your scale using the short-points of the dividers, which will now be at 1/18th of the total scale-length. Most proportional dividers have an adjustable tensioning screw/clamp to ensure stability of a chosen setting and thus accuracy when transferring measurements.
4 Reset the divider's long-points from the bridge datum to the now shortened scale – length (i.e. the full scale-length *minus* the first fret distance).
5 Without disturbing that setting, uplift the dividers and mark off the *second* fret from the first fret position again using the short-points.
6 Reset the divider's long-points from the bridge datum to the second fret position (i.e. the full scale-length *minus* the two first frets).
7 Without disturbing the setting, uplift the dividers and mark off the *third* fret from the second fret position using the short-points.
8 Repeat the process until enough frets are determined.

Readers will observe that the residual scale-length is being repeatedly divided by 18 as each fret is determined and marked off.

Smaller proportional dividers can work with large dimensions and scale-lengths that exceed the opening span of the legs. In that case, as with the Sector, it is necessary to work with a convenient fractional proportion of the length in question. Therefore, an

18” scale-length may be reduced to ½ or 1/3rd of its value (i.e. 9” or 6”) and the same method and procedure applied as above on the shorter scale-length. However, to determine the ‘full size’ frets of an 18” scale, the short-point dimensions derived must be doubled or tripled accordingly when marking out on the full size scale-length. For example: a first fret size, determined on a 9” long-point setting yields 1/2” at the small-points. That dimension would need to be ‘stepped out’ twice on a full size 18” scale-length to mark the first fret, but a 9” scale length would continue to be used for the subsequent measuring and conversion.

Results

Using the experimental dividers tool and the method described on a 600 mm scale-length, I achieved an accuracy of minus 1.00 mm to the ideal octave point (12th fret). Compared with a mathematically derived scheme, the other fret positions were easily within experimental error and good enough for an equal tempered scale with either fixed or tied frets. See **Table 1** below[9].

Conclusions

Equal temperament fret determination by the ‘Rule of 18’, is a straightforward and quick process using large proportional dividers[10]. Smaller dividers can be used by working with a simple fractional proportion of the actual scale length.

Compared with the Sector, the proportional dividers seem potentially less accurate generally due to possible user error, or imprecision of the instrument settings. The Sector is a more stable, precise tool capable of interpolating divisors to a finer degree and allowing refined adjustments to gathered data in real time.

On examining historical instruments with fixed frets, or fret marks, it is possible to check if the luthier applied the Rule of 18 consistently by measuring any fret size (i.e. the distance between any two adjacent frets) and comparing that with its relationship to the relevant portion of the scale length. For example: measure a distance of ***x*** mm between frets 3 and 4. Check if that value is 1/18th of the scale-length measured from fret 3 to the bridge. Repeat for other sample frets to verify any conclusions drawn.

We like to infer or discover what equipment and methods were used by early luthiers, but as yet we have no specific early documented evidence of how equal temperament fretting was generated practically, during instrument design and manufacture, or how widely the Rule of 18 was accepted and applied. However, this article and experiment demonstrates the Rule’s ease of practical application using even the simplest equipment and no special skills or maths.

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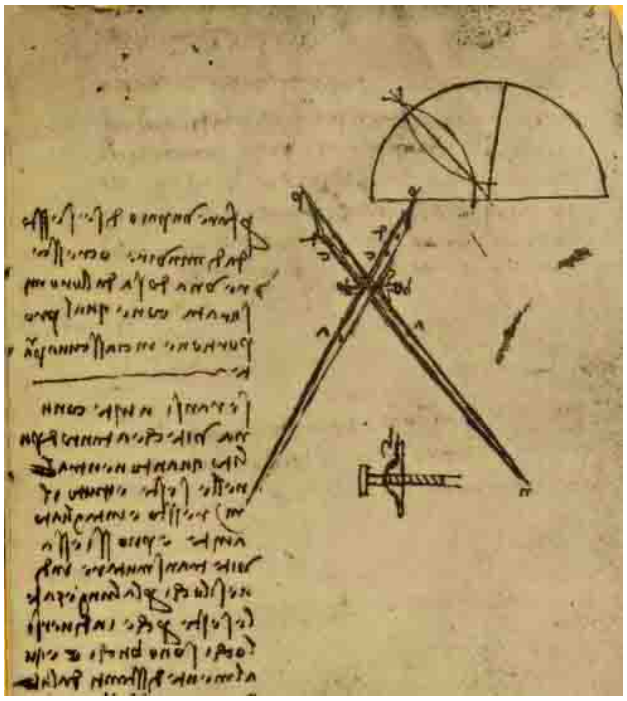

Plate 5: Leonardo's proportional dividers (see note 4)

Bibliography

Virtual museum 'Museo Galilei' has a large collection of historical compasses, Sectors and proportional dividers:
https://catalogue.museogalileo.it/indepth/Compasses.html

https://www.fleaglass.com/ lists and describes many historical scientific instruments (for sale) including proportional dividers, Sectors and other early calculating devices.

Pinette, L. (1955). 'Tangible arithmetic III: The proportional dividers' in *The Mathematics Teacher, Vol 48* (2), 91-95. http://www.jstor.org/stable/27954802

Bordas, C., Robledo, L., & Knighton, T. (1998). '*José Zaragozá's Box: Science and Music in Charles II's Spain*'. Early Music, 26(3), pp 391-413. ISSN 0306-1078 http://www.jstor.org/stable/3128699 .

NOTES

[1] Although some other authors have speculated that proportional dividers or the Sector were used for this purpose, they have not demonstrated how, or provided any experimental evidence to test the idea.
[2] In this article I use the term 'dividers', rather than 'compass' and 'compasses', or 'pantograph', terms sometimes used in contemporary sources. Proportional dividers have four 'points' and a central pivot that adjusts the pairs of points (referred to as 'long' and 'short' points) to the required proportion. The Sector is also sometimes described as a 'proportional compass(es)', or 'proportional dividers' in some museum catalogues, which may also confuse, but the Sector has a hinge/pivot at one end and looks much like a two-armed folding ruler.
[3] Chris Egerton '*Equal temperament fret position determination using a Sector*'. FoMHRI Quarterly 155. Comm 2161. A Lute Society affiliated publication. October 2021.
[4] Leonardo da Vinci *'Problèmes de Géométrie et d'Hydraulique…'*. Edouard Rouveyre, editeur, Paris 1901. Feuillet no. 4. p36 (See **Plate 5**). Digital copy sponsored by Getty Research Institute. https://archive.org/details/problemesdegeome01leon/page/n35/mode/2up accessed 7 August 2021.
[5] Euclid's Elements, Book VI, Propositions 4 and 5

[6] Vincenzo Gallilei (ca. 1525 - 1591) is credited with the 'Rule of 18', but opinions differ.
[7] D.J. Bryden *'Evidence from advertising for mathematical instrument making in London, 1556–1714'*. Annals of Science, Vol 49 (4), pp 301-336. Taylor & Francis 1992. DOI:10.1080/00033799200200281 https://www.tandfonline.com/doi/abs/10.1080/00033799200200281 Accessed 5 August 2021 All archives and current issues of Annals of Science are available online.
[8] Thomas Hood, Galileo and Edmund Gunter all published their own designs and manuals for the Sector between 1598 and 1623.
[9] Results, **Table 1:** Math derived vs proportional dividers derived fret positions from the nut in mm. 600mm scale length

Fret	**1**	**2**	**3**	**4**	**5**	**6**
Math	33.33	64.81	94.54	122.62	149.14	174.18
Expt.	33.00	64.00	93.50	121.50	148.50	173.25

7	**8**	**9**	**10**	**11**	**12**
197.84	220.08	241.28	261.20	280.02	297.80
198.00	220.50	241.50	261.50	281.00	299.00

[10] A twelve-fret sequence took about 15 minutes to generate.