FoMRHI Comm 2161 **Chris Egerton**

Equal temperament fret position determination using a Sector

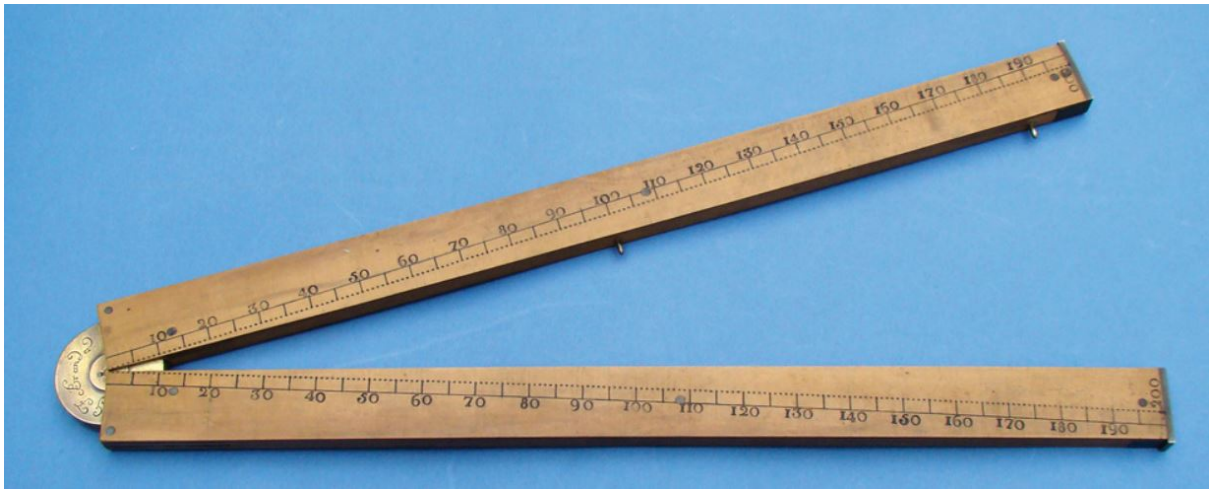

Plate 1: An 18th Century Sector for determining proportions of quantities. Size 65 cm long when fully open. Made of Boxwood and brass. Single scale with subdivisions. Note that the scales originate from the hinge pivot point.

Abstract

The 'Rule of 18' division is generally assumed to have been used historically for equal temperament fret calculation[1], at least as a basis, but there is no known early record of equipment or methods employed by musical-instrument makers for that purpose. John Dowland's method of lute fret layout[2] uses compasses and a complicated trial/error proportional subdivision of a scale length, but the method seems impractical and cumbersome, subject to inaccuracy and probably not useful to a working artisan in a dynamic manufacturing environment. A few fret position templates survive from Stradivari's workshop, but with no obvious indication of how they were derived.

I postulate the historical use of the Sector, sometimes known as a 'proportional compass', for calculating equal temperament (and other) fret positions. The Sector, even in a very simple form, makes short work of equal temperament fret calculation and requires minimal skill. I demonstrate the simple equipment and easy method.

Introduction

The Sector (**Plate 1**) is an early device used for calculating *proportions* of quantities, such as length, weight, area etc. without the need for complex mathematical work or involved geometric drawing by the user. In its simplest form virtually no skill is required to use a Sector, other than that of employing compasses/dividers to input and extract the required information. Early lute-makers obviously had no digital calculators or computers and we presume that most workshop artisans had only basic educational backgrounds. However, the workshop masters, who designed and

oversaw musical-instrument manufacture, surely had superior education, including training in maths and geometry, either through formal education, or passage through the apprentice and Guild system. Such an educated person could use a Sector with ease and could instruct a more lowly workshop artisan or apprentice on fret calculation in minutes.

Theory
Ancient Greek mathematicians and especially Euclid[3], proposed and demonstrated mathematical and geometrical proofs that were relied upon in later centuries to design and develop methods, means and machinery for the advancement of human society. Euclid knew and proposed that similar triangles (i.e. triangles with all the same (congruent) angles) had their like sides in direct proportion[4].

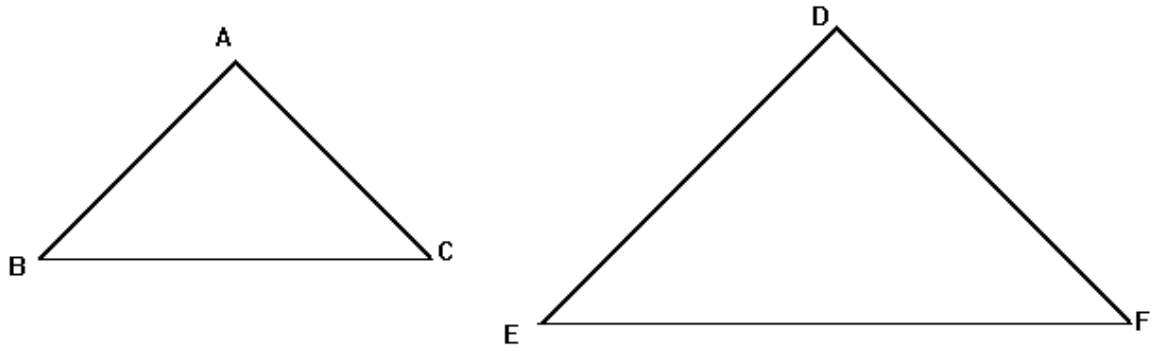

FIG 1: Two similar triangles

If, in the above,we take a point half way along AB and draw a line across, parallel to BC, that line is in proportion and exactly ½ of BC.
If our point was 1/3 along AB and a line drawn across parallel to BC, that line would be a 1/3 proportion of BC. The same applies to all other parallel lines drawn across. The lines would all be in relative proportion to each other and to their distance along AB.

This is the principle behind the Sector: a simple device for determining proportional relationships without the need for lengthy or difficult mathematical calculations.

The Sector, as an instrument, is simply two arms of such a triangle, say AB and AC above, hinged at point A, so that the angle between them can be varied (**Plate 1**). Each arm is marked with an equidistant scale of any convenient units, either standard units such as inches, centimetres etc; or just an arbitrary, but uniform division scale. Measurements can be set, or taken transversely across the 'like' points on the opposite scales using compasses or dividers.

History
Various theories exist about when or by whom the Sector was 'invented'. Galileo designed and had created his own elaborate Sector in c.1598, along with notes for his students and later a user manual[5]. Sketches in Leonardo's works indicate a device for obtaining relative proportions[6]. Architect Antonio de Sangallo the Younger, in his notes of c.1530, described and sketched a flat-armed proportional compass to '*... find any proportion whatsoever, of any kind'*[7]. Certainly the Sector, as a working tool, was known, eventually widely used and further developed in the 16th century and later, but I propose, that since the principle is ancient and proven, there were probably functional devices and certainly the method and principles, in use much earlier. Pyramids were built, fabulous architecture constructed and complex sailing vessels created in ancient history and it seems only logical that simple methods of determining proportions of size, weight, area or length were available to builders, designers and working artisans.

The more recent Sectors of the 18th and 19th centuries, of which many survive in old drawing instrument sets, have more elaborate scale markings and methods for all types of fancy calculations, but for our simple task of calculating equal-tempered fret positions we require only the most rudimentary Sector, with the simplest markings, that anyone could make and use and the simplest method that a child could learn.

Application and tools
For this article I made a simple experimental Sector similar to **Plate 1** using the hinge from a broken, discarded folding rule and some wooden strips (**Plate 2**). One could make a basic Sector using some thin cardboard and a punched eyelet, rivet or drawing-pin as a pivot. The two equal arms on my Sector are about 70 cm long, although any length can be used, preferably greater than the scale-length of your musical-instrument. Smaller Sectors, as found in old drawing instrument sets can do the job, but for this article the simplest procedure is described.
The arms should be marked (and numbered) with a regular equidistant scale of any convenient dimension. At least 18 equal divisions are needed and subdivisions can be introduced, but are not essential for a demonstration. If making your own Sector, just transfer a convenient scale directly from any standard rule, or 'step out' any suitably sized divisions using dividers.

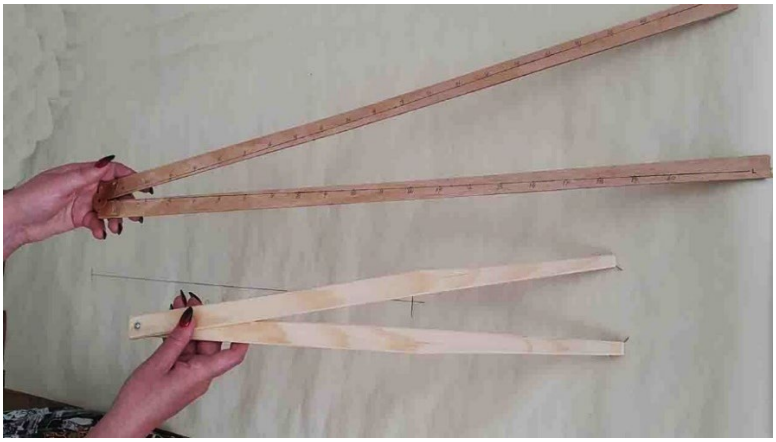

Plate 2: Sector (top) and compasses (bottom). User made.

Method
To use the Sector for fret position determination, first draw a straight line (scale line) on paper (e.g. on your working drawing) and mark off the intended scale-length. In our case it need not be a known measured length (i.e. of inches, centimetres etc) and the Sector user does not require any other information except the total length marked off on the scale line. The scale line will be marked up with the derived fret positions as they are determined:

1 Open a large pair of compasses, dividers or trammel points and set them to the scale-length of your fretted instrument as shown on the scale line.
2 Place one compass point on division mark '18' of one arm of the Sector and open it to place the other compass point across on the corresponding '18' division mark of the other arm. (**Plate 3**).
3 Uplift your compasses without disturbing the Sector setting.
4 Adjust your compasses or trammel points closer and take a measurement across the Sector scale markings at '17' on each arm.
5 Transfer that measurement directly to your paper scale line and mark the distance from one end of the line. That is the first fret position and marks 17/18th of the scale length from the bridge position or 1/18th from the nut.
6 Next, keeping the same compass/trammel setting you just obtained, adjust the Sector and place the compass points on the '18' scale markings of each arm again.
7 Repeat steps 3,4,5,6 until you have sufficient fret positions. For a typical lute, 8 or 9 frets are usual.
8 If desired, make a fret position template for the instrument for use during workshop manufacture or for future reference.

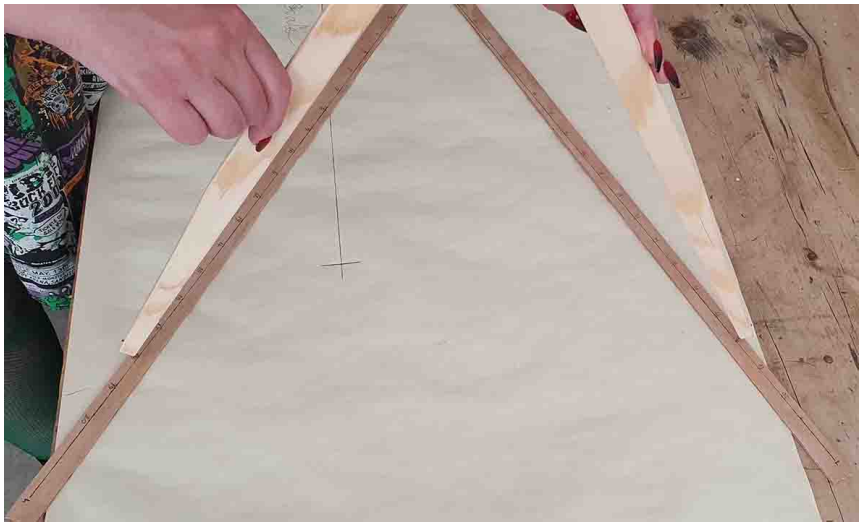

Plate 3: Transferring a reading to Sector marks at '18'.
The Sector should be on a flat surface to allow adjustment and accuracy.

If there are no suitable compasses or trammel points available one can use a narrow strip of card, stiff paper or thin wood having a straight edge and use a pencil to mark and transfer the measurements.

Readers will note that the process of reducing the '18' to '17' on the Sector and repeating the process as described is actually dividing the remaining scale-length again by 18 as the individual frets are determined.

Cross-checks on accuracy can be carried out if desired, because sometimes accumulated errors arise from repeated manual measurement transfer.
In the case of equal temperament fret 5 is at ¼ proportion of the scale-length, so using the Sector, open the compasses again to the full scale-length and place the points on the '16' marks of the opened Sector. Uplift your compasses and take a measurement from the Sector's '4' marks (4 is a ¼ proportion of 16) and check that the measurement corresponds to the 5th fret position on your scale line. Adjust its accuracy as necessary. Accuracy is more critical with shorter scale lengths.
Do the same for the 7th fret position (which lies at 1/3rd of the scale-length) using the marks at '18' and then '6' to check the fret position (i.e. 6 is a 1/3rd proportion of 18).
The 12th fret can be similarly checked by using the '18' marks and taking the measurement reading from the '9' markings on the Sector (9 is a ½ proportion of 18)

Conclusions
The use of a Sector is a very simple, accurate[8] method of fret position determination, based on the 'Rule of 18', for <u>any</u> typical equal temperament scale-length and without using mathematics or complex geometry.

The Sector was used from the 16th to 19th centuries as a general purpose calculator and a working tool in areas such as architecture, surveying, maritime navigation and map making, munitions and warfare, goods and commodities trading; solving real

world math problems when people's education was variable to say the least. Logically speaking, musical-instrument designers and makers from the 16th century and perhaps earlier, would have known of and likely used a Sector in their work.

When using a Sector simple fractions are useful to know, such as ½, 1/3 , ¼ of any given whole integer, but even those proportional relationships can be determined with a Sector without any maths at all, especially if the units of a quantity are unknown, or the length in question is an awkward fractional dimension.

It will be apparent to the wise that the method described here can be varied, further simplified, or made more precise and that fretting for other temperament scales can also be determined using a Sector. For example, the first fret position can be found using half the scale-length as a starting measurement, as fret one is at 1/9th proportion of half the scale-length. The division factor of 18 could be interpolated by using subdivided scale markings to improve overall precision, or to accurately modify certain fret positions as required in some other scales. A smaller pocket-sized Sector could also be employed by using a slightly different approach.

There has already been a great deal of research and theoretical discussion of temperaments, pitch and scales that I need not discuss in detail, but if there is sufficient interest in this topic of the Sector and how it was employed in our musical history, I will consider more articles. Meanwhile the bibliography and endnotes direct readers to some excellent material, including original texts, about the Sector and its general use as an historical calculating device.

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NOTES

[1] Vincenzo Galilei (1520-1591) is said to have originated the rule, but opinions differ.
[2] Dowland J. '*A Varietie of Lute Lessons*' 1610 edition
[3] Berlinski D. '*The King of Infinite Space--Euclid and his Elements*'. Perseus Books Group, New York City 2014.
[4] Euclid's Elements, Book VI, Propositions 4 and 5.
[5] Galileo sold Sectors to his nobility students.One example is here: http://waywiser.fas.harvard.edu/objects/3608/galileos-geometrical-and-military-compass?ctx=db91f8e0-de24-4fb3-ada5-c9d5cb6d0d81&. Viewed 22 July 2021.
See also Galilei, Galileo, '*Le Operazioni del Compasso Geometrico et Militare*', third edition, Padua 1649.
Scan of the original book available at the Internet Archive. Free English translation also available at https://brunelleschi.imss.fi.it/esplora/compasso/dswmedia/risorse/eleoperazioni.pdf Accessed 22 July 2021
[6] Codex Forster Vol 1, National Art Library,Victoria & Albert Museum, London. Museum no. MSL/1876/Forster/141/I . Available to view at hi-res online at 22 July 2021.
[7] https://kartsci.org/kocomu/computer-history/proportional-compass-sector/Antonio Sangallo Retrieved 22 July 2021
[8] Using the simple equipment and the method described I obtained accuracy of -1.5mm to the octave point of a 600mm scale (i.e. the twelfth fret point), Time taken to mark 12 frets was about 15 minutes.