The next issue, Quarterly 110, will appear in October 2008. Please send in Comms and announcements to the address below, to arrive by October 1st.

Fellowship of Makers and Researchers of Historical Instruments


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Like the Seven Sleepers of Ephesus, who fell asleep during the persecution of Diocletian and awoke 200 years later in the time of the Christian emperors to find the world somewhat changed, FoMRHI awakes from a slumber of six years to find that things have moved on. Virtually everyone reading this will by now have a computer and the ability to send and receive PDF attachments, and may well be a member of various internet discussion fora; and most of us now regularly search ever-expanding internet resources for any bits of organological, historical, musicological, iconographic and codicological data which elude us. In the field of organology and musicology research things have progressed too. In my own small field of lute studies, some significant old music manuscripts have been discovered since 2002, and the work of producing facsimiles and modern editions proceeds apace. Music typesetting technology is continually being improved. String makers have continued to develop and improve historically informed strings, and significant historical researches have been published which shed light on such matters as silk, twisted, roped and loaded lute strings. Historical instruments have been examined, measured and restored. I have no doubt that similar progress has been made across the whole field of musical instrument research. So FoMRHI is relaunched in the confident hope that members will have six years-worth of pent-up insights, discoveries, thoughts and queries that they wish to share with fellow researchers and enthusiasts all around the world.

The intention is to pick up pretty much where we left off. There are unspent subscription monies in the bank, so existing members will receive the first few issues of the Quarterly free of any further charges. As an encouragement to contribute, Comms from non-members will be rewarded with a free year's subscription.

A number of people have asked whether we should not go over to electronic formats. In fact, notwithstanding the generalisation made above, a significant number of subscribers and contributors still neither have nor even want to have a computer. There would also be the problem of ‘free riders’ or unauthorised sharing if the Quarterly was sent out electronically; and for sheer handiness of use, in the workshop, on the train, or indeed in the bath, the A5 format is still hard to beat. In these days of eco-consciousness, an A5 booklet, with pages printed on both sides, still uses up less woodpulp than the printing out of A4 pages from your computer (few people, I imagine, would actually sit and read 80 pages of text from their computer screen.)

Nonetheless, there are plans to set up a website, and ultimately to scan in all Qs to date, in downloadable or screen-readable format. This should help put us back on the map, and really kick-start our revival.

This first 'revived' issue is different from the normal format in that it is in effect a monograph on the physics of lutes by retired physicist Chris Coakley. He sent his material in before anyone else and so it appears on a 'first come, first served' basis! We have nearly enough material for another FoMRHI/Q in hand, however, this should appear in October. The next deadline for Comms will be the 1st October.

It goes without saying that the revival of FoMRHI will only succeed if YOU, YES YOU, send in contributions whenever it occurs to you that you have something to say. Comms which pose a question are just as welcome as those which present an answer. Members' announcements are also welcome – if using a computer, please send these as plain text emails, rather than attachments.

A stall at the Greenwich Exhibition? We are currently discussing the possibility of a FoMRHI stall at the Greenwich International Early Music Exhibition, of 14th-16th November. This may or may not happen, but please let the Secretary know if you could come and help staff the stall. For an hour or so of help, you get in free!
Where are they now? Over six years our address database has got a bit out of date. Does anyone have current contact details for the following? Please let us know if you can give us current addresses for: Eric Chapman, Andrew de Witt, Bernard Emery, Alan Higgitt, David E Williams, Philippe Beltra, Joseph Pachschwoell, Dmitry Badiarov, Jan Hermans, Peter Wilkinson, Hans von Busch, Renata Keller, Martin Christian Schmidt, Saulius Steponavicius, Daniel Papuga, Anders Emmerfors, Thomas C Boehm, Wesley Brandt. They paid their subscriptions along with everyone else, and are entitled to receive FoMRHJQ.

Email addresses, please! If you haven’t received any emails from us this year, that means we don’t have your email address. It makes communication so much easier if we have it. We promise not to send out any spam, or pass it on to anyone else. Please send a brief message to Lutesoc@aol.com, and we can add you to our list.

MEMBERS' ANNOUNCEMENTS

Classical handmade guitars and one steel string. At cost price. Spruce fronts, indian rosewood backs and sides, ebony fingerboards, oiled finish. Made with great enjoyment by me. Call Mike, on 01702 556892 (Southend area).

Jan Bouterse’s dissertation about Dutch woodwind instruments and their makers, 1660-1760, has been translated and published in English language at: http://home.hetnet.nl/~mcjbouterse/inhoud&samenvatting.htm; also http://www.kvnm.nl/current/03Catalogus/BN_9.htm, with English summary. He is on the point of finishing a (thick) manual on making woodwind instruments, incorporating his experience gained from research into historical instruments, though this is currently in Dutch only.

Contact: Jan Bouterse, Sandenburg 69, 2402 RJ Alphen a/d Rijn, Netherlands tel. + 31 172 445957 e-mail: mcjbouterse@hetnet.nl
STANDING CALL FOR PAPERS

The Fellowship of Makers and Researchers of Historical Instruments welcomes papers on all aspects the history and making of historical musical instruments. Communications or 'Comms' as they are called, appeared unedited (please don’t be libellous or insulting to other contributors!), so please send them EXACTLY as you wish them to appear – in 12 point type, on A4 paper with a 25mm or 1 inch border all round, or to put it another way, if you are using non-European paper sizes, then the text area must be 160 x 246 mm (or at least no wider or longer than this). Our printers usually make a reasonably good job of scanning photos.

You can send contributions EITHER on paper, OR as a Word-compatible or PDF attachment. If you really do not have access to a word processor of any kind, we may be able to retype typed or handwritten submissions.

NOTE OUR NEW ADDRESS:

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and the email address for Comms sent as attachments (and other email correspondence) is Lutesoc@aol.com

Non-members will be given a year’s free subscription if they send in a Communication to the Quarterly.

If you ever sent in a paper (in the last 6 years) for the Quarterly, and it never appeared, please re-send it, to the new address.

There are plans to scan back issues of the Quarterly and make them downloadable from a website, to be set up; in the meantime you can obtain back issues for the princely sum of £3 per issue, including postage; send a cheque payable to FoMRHI, at the above address, or write with your credit card details.

If your interests have changed, and you don’t now want to be a member of FoMRHI, please let us know, to save postage costs.
TUNING TEMPERAMENTS FOR LUTES (1)

This communication is the first of seven on the science of lutes. Four main papers were initially hand-written, mainly for personal ordering of leisurely theoretical investigations. They were made available to Lute Society members, and the Secretary has suggested that this topic of acoustics would be of interest to FoMRHI. Several further papers, and letters to Lute News, have been added to cover many aspects of physics for lutes. The seven communications are referred to in the text as paper 1 etc, after the numbers in the titles.

A GENERAL INTRODUCTION

It may be helpful, at the start of this first communication, to give a general introduction to all the various subjects, types of approach, and the form of the presentations. This will also serve as an abstract of some results and conclusions. The methods have been the two extremes of physics theory, and a little practical experience in making, tuning and playing, rather than a more usual middle road of detailed testing. A full concurrent experimental study would have been ideal, but this would have required a laboratory. Only a few tests on strings and soundboard materials were possible. However, each paper, after this first one on temperaments, treats an important practical issue for which no previous solution appears to be available in a textbook or research journal.

Musicians and instrument makers may be surprised to know how little understanding there is in the science of musical instruments. Standard treatments for ideal vibrating strings, membranes, plates and air columns, were made by physicists and mathematicians over a century ago. Two notably successful explanations are the basic frequencies of strings and drums. More difficult cases are wind instruments, where there are complex end effects at the bell, and especially the mouthpiece. For lutes, and also related guitars, violins, harps etc, the soundboard has features such as the bars, bridge and attached body, which produce a system very different from a simple plate. In addition, real lute strings can deviate significantly from the ideal case, and this can affect sensitive aspects of tuning and sound quality. The present work, over the last four years, started practically with fitting frets and strings to a new lute. Pure curiosity about tuning temperaments led to string properties, historical strings, then soundboard acoustics, strengthening by the bars, and the overall system.

(1) This first communication is a rigorous derivation of tuning or temperament systems suitable for lutes. This was a completion of a much earlier attempt to find consistent schemes for keyboards. I later confirmed that this is a general form of 'meantone', allowing understanding and calculation for any interval and degree of tempering. Clearly the system is not new, but the paper could be a helpful addition to available lists of fret positions, a concentration on the major third, and tuning meters. A practical letter follows on how temperaments can affect consorts of differently pitched lutes.

(2) The second communication begins with a letter, presenting a new quantitative criterion for the elastic modulus of string materials. It has long been recognized that bass strings need good elasticity, and this work grew out of practical concerns with tuning by frets, and tone. This leads to the second main paper, which attempted to deduce the properties of ancient lute strings, none of which may survive, from the unusual feature of angled bridges on old lutes. Several
explanations had been considered previously, and I hoped a firm resolution would be possible using the elasticity analysis. The requirements were found to be so extreme and variable, as to be impractical, and unlikely for an explanation. The only other physical alternative was seen to be a natural tapering in the mass distribution for the old gut strings. This paper ended with an analysis of taper, but at that time a difficulty in deciding the relative roles of elasticity and taper for old strings. The following section gives details of the elasticity theory, which is useful for strings in general. A final letter suggested a new synthetic material for modern bass strings.

(3) For the third communication, a year after the soundboard research described below, the further detail of angled necks, and a close reading of written sources, led to new and surprising conclusions about taper, possibly providing a satisfying conclusion to the string research.

(4) From the start, a distant goal was the lute soundboard and body, and this is the subject of the fourth to seventh communications. The initial focus was the bridge, through which the strings act, but the presence of the bars appeared to prevent low frequency resonance. This obstacle was eventually removed by devising a new analysis for a combined vibration of bars and soundboard. Several series of resonant modes could then be predicted, depending on the bar sizes and their spacing, and the soundboard properties. This appears to be entirely new, even for modern instruments, and these ideas can also be applied to other string instruments. This research appeared in the fourth main paper, with added sections on guitars and vihuelas, and forms the fourth communication.

(5) For the fifth communication, the notion arose for an influence of string tension on the soundboard resonances. In turn, this work led to a new way of understanding and predicting the structural stability provided by bars. This is the primary purpose of bars, and the analysis also places useful limits on the acoustic possibilities. Some related theory gave an explanation for the striking concave or hollowed rib surfaces on lutes.

(6) In the sixth communication, the sensitive area around the bridge is revisited, with an analysis of changes to the resonances caused by the bridge, the small treble bars and the bass bar. This led to an explanation for the slight angling of bridges and main bars seen on several old lutes. Just as angling had been a puzzling spur in the string work, it was also an important question for the soundboard theory.

(7) The seventh communication combines together all the preceding analyses to see how various sizes of lute might be optimized, or even designed from first principles, with due regard to the basic outline, original materials, purpose and human action.

As an overview, the research on soundboards gives new basic results for both modern and old instruments, and then concentrates on the known design of old lutes and some related instruments. In contrast, the work on strings extends some important details for modern and old strings, then tries to deduce properties of old strings from the design of old lutes and writings. This has been motivated by the sheer curiosity of applying physics to a natural phenomenon – an ancient masterly construction from dead biological materials. We all tend to think of instruments and even modern types, in a very personal, qualitative, tactile way. It is therefore doubly strange trying to analyse these instruments, invented and perfected in an age before Newtonian science, which must now be the basis for a correct understanding. Much could be said about the possible relations between physics, mathematics, instrument making, craftsmanship, engineering, experiments, materials, and also aesthetics. The present work may be most suitable to those with scientific curiosity, but it may also interest historians and inquisitive players. A good use for makers, similarly to practical engineers, may be understanding and explanation, rather than any additional detailed designs.

The presentation follows the initial order of the research and writing, with few later revisions. This gives a correct impression of work in progress, rather than an ordered treatise expressing a final opinion. In the second communication, or paper 2, short notes (N.B.) refer to later
conclusions in paper 3. The order of subjects is almost opposite to making a lute, since deeper
topics took time. Some reading in reverse order may be useful, the major subjects being overall
design, structural barring, soundboard resonances, string taper, elastic theory, and	temperaments. Only a few equations have been included from the original workings, and they	always lead to practical conclusions. However, derivations of new results are outlined fully, and
could be supplied. An increasingly important aspect of the research was making physical	progress where exact mathematical solutions are not possible, and where standard treatments in	acoustics may not be entirely relevant to understanding instruments.

For specialist readers, it was thought unnecessary to summarize details of lute history and
design, and I am not highly knowledgeable here. However, it is worth noting that most surviving	lutes are the baroque types after 1600, and examples of the earlier renaissance lute are much	scarcer. Instead of several formal introductions and conclusions, the many added section	headings should be a good guide, but there may be a shortage of diagrams. This first personal
typing venture has been encouraged and assisted by my wife, and the local museum. I am also
grateful to many society members for advice and interest in this object of fascination.

TUNING TEMPERAMENTS

As a newcomer to fretted string instruments in 2003, I was introduced to descriptions of frets for	'sixth comma meantone' tuning (Refs 1 and 2). Having previously met 'meantone' for tuning	perfect thirds on keyboards, using beats but without basic theory, I have tried to construct	consistent tuning systems for lutes. I suspect this is a general case of meantone, but it can be
to understand and try out their own temperaments, or possibly write programs for tuners,
fingerboards or keyboards.

1. The Problem

A tuning with some perfect or natural intervals is listed in Table 1. The frequencies of all notes	are shown, relative to the reference C = 1. The details are based on 'difference tones', or large
beats such as ½, ¼ etc, and include different semitone intervals. However, if we use these notes
to set up a new scale, in F say, only some correct intervals are preserved. For example, D = 9/8
when relative to F = 4/3 produces a sixth at 27/16, which is far sharper than a perfect 5/3. These	problems increase for further keys and, since the simplest piece in C requires chords on other
notes, better tuning systems are needed for instruments with scales of about twelve fixed notes.
The problem can be expressed formally by number theory. Cycles of twelve perfect fifths
C, G, D, A, E, B, F*, C	extsuperscript{*}, G	extsuperscript{*}, D	extsuperscript{*}, A	extsuperscript{*}, E	extsuperscript{*}, B	extsuperscript{*}, etc
can never return exactly to a C, since no power of 3 can equal a power of 2. Seven octaves	higher

\[
(3/2)^{12} / 2^3 = 531441 / 524288 = 1.01364
\]

This is a tiny frequency interval, between a quarter and a fifth of a semitone, known as a
Pythagorean comma. An alternative expression is that a sequence of four perfect fifths cannot
give E as a perfect third. Here the third is wider than perfect by an amount

\[
(3/2)^{5} (4/5) / 2^2 = 81/80 = 1.0125
\]

which is a syntonic comma (strained tightly, severe). The example D above has this degree of
sharpness, and is called a Pythagorean sixth, (3/2)^3/2.

2. Even Temperaments

Although it is well known that equal temperament, ET, offers a good solution to many tuning
problems, the question arises whether other consistent tuning systems can be devised for some,
if not all, keys. The effect would be to push problems of the comma into unused keys.
The easiest possibility is to try setting up a scale, which can preserve identical but slightly imperfect intervals over the common keys. We can take a third as say \(\alpha\), a fourth as \(\beta\), and a fifth as \(\gamma\). Now we set up a sequence of fourths. This interval is theoretically basic as well as practical for lutes, as seen later.

\[
C = 1, \quad F = \beta^2, \quad B = \beta^4, \quad E = \beta^3/2, \quad A^\# = \beta^{\frac{3}{2}}, \quad D^\# = \beta^2/4, \quad G^b = \beta^\frac{5}{4}, \quad C^b = \beta^\frac{7}{4}, \quad \text{and further} \quad F^b = \beta^{\frac{8}{5}}, \quad B^b = \beta^\frac{9}{8}, \quad E^b = \beta^{10}/16, \quad A^{\#b} = \beta^{11}/16, \quad D^{\#b} = \beta^{12}/32, \quad G^{bb} = \beta^{13}/32. \quad \text{etc.}
\]

Here the divisors 2, 4, etc will place notes within the octave \(C = 1\) to 2. This amounts to requiring perfect octaves, and is the only constraint. Next, a series of descending fourths, or ascending fifths with \(\gamma = 2/\beta\), gives

\[
G = 2/\beta, \quad D = 2/\beta^2, \quad A = 4/\beta^3, \quad E = 4/\beta^4, \quad B = 8/\beta^5, \quad F^* = 8/\beta^6, \quad C^* = 8/\beta^7, \quad G^* = 16/\beta^8, \quad D^* = 16/\beta^9, \quad A^* = 32/\beta^{10}, \quad E^* = 32/\beta^{11}, \quad B^* = 32/\beta^{12}, \quad F^{**} = 64/\beta^{13}, \quad C^{**} = 64/\beta^{14}, \quad G^{***} = 128/\beta^{15}. \quad \text{etc.}
\]

The double flats and sharps are given for completeness, and to show later effects with octaves and enharmonics. These formulae are listed in Table 2.

It may be familiar, and clear from musical theory, that the series of fourths or fifths generate all the notes. If not, then I also showed rigorously that these formulae are the solution to a set of equations requiring the same intervals, and using exactly the same notes, for several scales. Since all these notes are integral powers of \(\beta\), or \(\gamma\), it follows that no other interval could be used to generate the notes. For example, using the third, \(\alpha\), gives: \(C, E, G, B, D, \) or \(C, A, F^b, D^b\).

However, all the formulae can be expressed in terms of another interval, such as \(\beta = (4/\alpha)^{\frac{1}{14}}\).

The major third is of particular interest, with \(\alpha = 4/\beta^4\). This can be seen as a relaxation of the formula for the syntonic comma, now giving a simple or perfect relation between imperfect intervals. This shows why the fourth is theoretically significant, and it is practically important later for tuning open lute strings. Many other relations and symmetries are touched on later.

3. Practical Scales

Now we need to be more practical and choose just twelve notes for a lute scale, since frets for two enharmonics would not be generally necessary, possible or convenient. Lute music and technique is biased towards flat keys, so the scale based on \(C\) will use \(C, D^b, D, E^b, E, F, G^b, G, A^b, A, B^b, B, C\)

which has just naturals and flats. Table 3 lists the note formulae for this scale. All the intervals in this scale are identical for the six major keys of \(C, F, B^b, E^b, A^b, D^b\), by musical theory and directly verifiable from the formulae. Outside this range, for \(G\) major an \(F^b\) is missing but all other intervals are preserved, included the third, \(B\). Similarly, for \(G^b\) major we lack a \(C^b\) but have the third, \(B^b\). For flatter keys the third and tonic are absent. For \(D\) major we lack the third, and progressively more notes for sharper keys. For minor keys, sharps are missing for \(A, D\) and \(G\), but we have minor thirds, and also for \(E\) and \(B\). Only the three minor keys of \(C, F\) and \(B^b\) are completely consistent, since for \(E^b\) we lack \(C^b\). Hence, this tuning system can be equally good, or bad, for six major keys and eight minor thirds, and three to six minor keys and eight minor thirds. Beyond this region, accuracy and consistency decrease progressively. This might be a great improvement on a single exact scale, and perhaps ET, depending on the closeness to perfect intervals for suitable values of \(\beta\). The system might be called even temperament, since it is not necessarily ET.

4. Practical Instruments

The above system would suit keyboards with independent notes, and could be shifted to choose sharper keys. On a ‘G’ lute, the scheme could be used for positioning frets on the \(C\) string. With standard parallel frets exactly the same series of intervals occurs on each of the other five basic strings. To preserve overall consistency, the open string notes \(G, (C), F, A, D, G\) must be identical with those produced on the \(C\) string scale. (It is easier to keep \(C\) as the standard than
change now to G.) Hence, we find the relation $\beta^2 \alpha \beta^2$, or $\alpha \beta^4$, = 4 also holds for the open strings, or across a single fret. Now the G string scale has all the pitches on the C string, except for F from B on the C string. This follows from musical theory and can be checked from the formulae in Table 2. The F string has a C, from G on the C string; the D string has F, and C, from E and B; and the A string has F, C, and G, from A, E and B. The lute can therefore produce a matrix of several Naturals and Flats, with one further flat, and seven sharps mostly on the high eleventh fret. These eight exceptions could cause problems for flat keys, but possibilities for minor keys and sharper keys. Several detailed playing and musical effects are discussed in the references.

5. Examples of even temperament
Numerical examples are given now for this consistent tuning system. Table 4 uses different values of the fourth, $\beta$, to construct seven examples of twelve note scales. Columns 1 and 7 take extreme values for $\beta$ of 1.330 and 1.340. This looks narrow but it will bracket the entire range of practical interest. Table 4 uses only three decimal places to show the variations more clearly.

**Perfect fourths and fifths**
Column 2 is a scale based on a perfect fourth, where $\beta = 4/3$ or 1.3333. The fifth and major second are also perfect or natural. Perfect intervals are underlined in Table 4. The major third is $\alpha = 4/\beta^4 = 1.266$, which is very wide compared with a perfect 1.25 or 5/4. This basic scale is known as Pythagorean, and examples were noted in section 1. In the numerical results we only need to compare fourths, major thirds, seconds and sixths, since perfect octaves produce reciprocal effects in fifths, minor sixths, sevenths and thirds. Column 1 is now seen to be even less practical, having narrower fourths, and wider fifths and thirds.

**Equal Temperament**
Column 3 has a fourth with $\beta = 1.3348$, which is slightly wider than perfect, and this gives equal temperament (ET). This is a special case and has interesting properties in addition to the well-known ones. It arises when the cycle of fifths or fourths reaches a B equal to C, so there is no possibility of multiple sharps and flats (see Table 2). The condition for ET is that $\beta^{12} = 32$, or $\beta_e = 1.3348$. Here the enharmonic sharps and flats converge to common values. For example, C = D#, so that $8/\beta^7 = \beta^3/4$. Table 3 lists all the semitone intervals for a general $\beta$, which shows alternate values of $\beta^{3/4}$ and $8/\beta^7$. Hence, for ET all the semitone intervals are equal to $\beta_e^{3/4} = 2^{10/2} = 1.0595$, which is the usual definition of ET. For $\beta$ less than $\beta_e$, as for a Pythagorean scale, the sharps are sharper than the flats, as seen from the formulae for $C^\#$ and $D^b$ above, and also Table 5. For $\beta$ greater than $\beta_e$, the sharps are flatter than the flats, which is the more familiar case for practical temperaments.

There is a further property of practical interest close to ET. Table 5 lists all 21 notes, including enharmonics, for the extreme values of $\beta$. An exactly intermediate $\beta = 1.335$ is also listed, and this shows a substantial separation of enharmonic pairs, comparable with the variation of thirds and fourths over the extreme range of $\beta$. This occurs very close to ET with $\beta = 1.3348$, and it is a very sensitive effect. It suggests that even very good practical ET tuning may well give discrepancies, and audible differences between keys.

Table 5 also gives average values of frequency ratio between the extremes. These averages are very close to the accurate values for $\beta = 1.335$, particularly for the Naturals and Flats because these are smaller powers of $\beta$ than the sharps (see Table 2). This shows that we can interpolate, or take proportional values on a straight line between the extremes, in order to find sufficiently accurate fret positions for any chosen value of $\beta$.

The main intervals for ET are a slightly flat second and fifth, or sharp fourth, and a rather sharp sixth. The main problem, or price, for all these advantages is the wide third $\alpha = 1.260$, especially
for early music. The cause centres on the relation $\alpha = 4/\beta^4$, which shows that to narrow the thirds in columns 1, 2 and 3 we must further widen the nearly perfect fourth. This formula also predicts the sensitivity of these adjustments. If the changes in $\alpha$ and $\beta$ are $\Delta \alpha$ and $\Delta \beta$, then the fractional change in $\alpha$ is $\Delta \alpha / \alpha = -4\Delta \beta / \beta$, or $-\Delta \alpha = 4\Delta \beta$. This means the narrowing of the wide thirds is about four times the widening of the fourths, or narrowing of the fifths, which looks a good bargain. Similarly, the earlier large separation of flats and sharps can be derived as $(\text{flat/sharp}) = 1 + 9(\beta - \beta_e)$, which requires the use of a very accurate $\beta_e = 1.33483$.

**Perfect thirds**

Column 5 of Table 4 uses a value of $\beta$ which gives perfect thirds with $\alpha = 5/4$ or 1.25. The relation $\alpha \beta^4 = 4$ gives $\beta^4 = 16/5$ or $\beta = 1.3375$. The fourth is now slightly wider, and the fifth narrower, than ET, in line with the $\Delta \alpha$ relation. Also the second is flattened further, but the sixth is less sharp.

Column 6 shows an even larger value of $\beta$ giving perfect sixths, or minor thirds, with intervals 5/3 and 6/5. From the formula in Table 3, $\beta^3 = 12/5$ or $\beta = 1.339$. The major thirds are now narrower than perfect, and all the other deviations of column 5 have increased.

### 6. Practical tuning systems

All the above results show that the widest area of practical interest is from $\beta = 1.3333$ for perfect fourths, fifths, seconds and minor sevenths, up to $\beta = 1.3375$ for perfect thirds and minor sixths.

Larger values for perfect sixths and minor thirds are perhaps too extreme. The wider region contains all the perfect intervals, as seen in Table 4, which also shows the other perfect flats. If the main aim is to obtain better major thirds than ET, with its attractions of general simplicity, uniformity, and common usage, then the range of interest narrows to $\beta = 1.3348$ to 1.3375, or ET to perfect major thirds.

Column 4 uses an almost average value of $\beta = 1.336$ and produces values for all the intervals midway between columns 3 and 5, as expected by interpolation. This gives slightly narrow seconds and fifths, and wide sixths and fourths, but nearly perfect sevenths, $B$, and flattened fifths, $G_b$, in Table 4. Most importantly, the slight increase in $\beta$ from ET (by 0.0012) has halved the sharpness of the major third (by 0.005) to $\alpha = 1.255$, in line with $\Delta \alpha$ formula, which is a subtle compromise. In summary, better but not perfect major thirds have been obtained at the expense of the versatility of ET, but without losing reasonable fourths, fifths, and seconds, and gaining better sixths and sevenths. This improvement only holds for the keys and notes given above in sections 3 and 4.

Later it will be seen that fret positions from column 4 are close to those quoted in Refs 1 and 2 for a system known as ‘sixth comma meantone’, and hence labelled as SCM. I suspect the reason is that $\beta = 1.336$ looks equivalent to 1.3333 (1.0125)$^{1/4}$. Similarly, $\beta = 1.3375$ may be 1.3333 (1.0125)$^{1/4}$, so that perfect thirds are ‘quarter comma meantone’, or simply meantone. The present system gives complete scales for any chosen value of $\beta$. This might be called ‘x comma meantone’ where $x = \ln(3\beta/4) / \ln(81/80)$.

### 7. Fret positions

The derived frequency ratios for musical intervals, are easily converted into fret positions measured from the nut. The frequency of an open string with a length $L$ is proportional to $1/L$. For a fret a distance $l$ from the nut, the frequency varies as $1/(L - l)$. Hence the frequency ratio, $r$, for this fret is $L/(L - l)$. The relative fret positions, for any string length, are therefore given by $l/L = (r-1)/r$. Table 6 converts values of $r$ from Table 4 into relative fret positions $l/L$. This can be applied to any lute and, as before, interpolations are quite accurate. Alternatively, exact calculations can select a new $\beta$, or use sharps instead of flats. For further practical use and insight, real fret positions are given in Table 7 for a typical $L$ of 600mm. The diagram shows this, with the alternate widened and narrowed spacings, except for E to F. The opposite, but
slightly asymmetric, sharps and C\textsuperscript{b} are also shown. (Tables 6 and 7 actually used the three decimal accuracy of Table 4, so 0.1mm accuracy is not assured, or necessary.)

For anyone familiar with ET and its progressively smaller fret spacings it may be alarming, after learning that temperaments are only small changes, to see the large shifts in position for say the first, third, sixth and eighth frets. The differences accumulate with each successive fourth or fifth, and these frets correspond to 5, 3, 6 and 4 fourths, relative to the open string.

For practical frets and strings, it is useful to make small corrections of about a string diameter at each end. Lengths also need to be measured from the supporting loop at the bridge. These effects are largest, at about 1 to 2mm, for high frets but decrease towards the nut.

8. Suggestions for use

Most players probably use ET, and some may have adopted 'sixth comma meantone', both of which have some historical basis (see Refs). I initially set ET and managed to move the frets to SCM, but further movement to perfect thirds or sixths would not be easy. There would have been a similar but opposite change to perfect fourths, but perhaps not advisable after SCM. This Pythagorean tuning could be effective for earlier music, and it may be surprising to see that ET is almost exactly midway between Pythagorean and SCM. Players may like to try the extremes of perfect fourths, thirds and sixths, especially if they have spare lutes, with necks that are not strongly tapered, or more elastic frets.

If alternative enharmonics are wanted, such as F\textsuperscript{#} for G\textsuperscript{b}, C\textsuperscript{#} for D\textsuperscript{b}, or C\textsuperscript{b} for B, which are large differences, then Tables 2 and 5 are needed. All the formulae given above could be written into a simple computer program, requiring a single value of \(\beta\) for each temperament. However, interpolations from the tables are perfectly sufficient. If the only requirement is fret position, then the diagram provides everything, when scaled for string length.

If electronic tuning meters are used, some care with accuracy is needed in converting fundamental frequency ratios to the logarithmic scale of cents. The required deviations from ET, as used on a meter, can be calculated for any temperament, and accounted for directly during tuning. Even elaborate meters have only a few preset temperaments, but some may be programmable.

9. Further comments

The need to obtain notes with identical pitches on different strings, rules out most temperaments used for the twelve independent notes on a keyboard. My initial requirement for a consistent system, leading to a fixed ratio between fourths, has been fortunate for this later application to lutes. In the next section, imperfections in modern strings and frets can alter this, but not in a way which would produce useful changes. For example, a major musical problem on the F string is a G\textsuperscript{b} rather than an F\textsuperscript{#} on the first fret, and a small extra fret is sometimes used. The sixth fret has similar but lesser problems.

Perfect third meantone can be effective on early keyboards, but may have been unsuitable for lutes. Possible reasons are the 'eight exceptional notes' across the strings, as in section 4, and an alternation of extremely wide and narrow fret spacings for the left hand (see diagram). The practical effect of neck taper may have limited the flexibility of historical tunings, giving some difficulty in moving frets between SCM and ET, or further. Fixed frets on old instruments, such as citterns, might be good evidence. Pitch pipes, organs and various wind instruments could also be helpful. Historical and present tuning appears to centre on small variations around ET and SCM, which can be explored with a series of \(\beta\) values or simple interpolations of the lists or diagram.

In summary, on moving away from ET and some central keys there is an increasing possibility of meeting one of the 'exceptional notes' or another wrong enharmonic, so that sweeter thirds may be sprinkled with occasional greater sourness and dissonance.
10. Related physics for lutes

Formulae for the frequency of vibrating strings can be set up for homogeneous materials and
wound strings. Interesting phenomena can be explained, such as strings of different diameter
breaking at the same frequency. Practical measurements of string tension could be useful.
Any effects of pressing a string down to a fret can be analysed. The details will be complex, and
depend on lute design, but since they can affect fret positions, some indication is given here.
Geometric changes in sounding length can be calculated as negligible. The major effect is
stretching a string, which increases its tension and hence the frequency. Some compensation will
require longer sounding lengths, or reduced fret distances \( f \) from the nut. Briefly, even for a low
action height, \( h \), between string and frets, a first fret may need to be about 0.5mm nearer the nut,
with the correction decreasing for the higher frets. For larger action heights of about 3mm near
an eighth fret, the 0.5mm could apply to all frets. For higher actions, the corrections are
expected to increase strongly as \( h^2 \), and would make tuning and playing difficult. These effects
apply to all temperaments, and are additional to the earlier end corrections. The lower strings are
more strongly affected, so that frets might need slanting slightly, by about 0.5mm, towards the
bass nut.

Estimating these stretching effects, required an effective elastic modulus. This was measured, by
putting a small marker on a string, as 1mm of stretch producing a frequency rise of order one
semitone or a 12% increase in tension. Since the diameter of a peg is about 5mm, one semitone
also requires about 20\(^\circ\), or one eighteenth, of a turn. This means that 1cent of fine-tuning, with a
frequency ratio \( r \) of 1.0006, needs 0.01mm or 0.2\(^\circ\) of peg adjustment. This explains the
frustrations of tuning the many open strings on a lute. In contrast, positioning frets is much less
sensitive. One semitone corresponds to say 20mm, so that 1cent requires 0.2mm accuracy in fret
position. This is twenty times less sensitive than peg turning.

The uncertainties in estimating these corrections mean that great precision is not required in
calculating fret positions. Other unrecognized factors might also be present, so that a final check
with a meter or a good ear is necessary. However, it is useful to understand the dependency and
direction of these effects, and this work shows that detailed calculations are possible.
Related aspects of evenness of touch across strings of various diameters and tension would be
more difficult. These, together with fine-tuning, will depend on individual perception. The
sound quality of various string materials, and lutes in general, is a great challenge for physical
theory, and experiments. Finite element methods on a computer may seem promising, but
problems might be anticipated with the very different resolutions needed for strings and a lute
body.

N.B. These early conjectures developed into the next six communications.

11. Numerical thoughts

The cycles of fifths can become very nearly perfect after 50 cycles over 351 octaves \((50\times7+1)\).
The first cycle is sharp by a Pythagorean comma, 1.01364. This should really be regarded as a
marvel, which has permitted many traditions of music, as much as a nuisance for tuning
instruments. After 2 cycles the sharpness is 2 commas, and after 50 cycles the sharpness is a
factor of about 2, or the initial note an octave higher

\[
\frac{(3/2)^{100}}{2^{351}} = 1.000031
\]

Later, I found that ancient Chinese theory was based on five cycles of perfect fifths. The note B,
five fifths above an initial C, after sharpening by a further four commas is very close to C three
octaves higher

\[
(3/2)^5 \cdot (1.01364)^4 / 8 = 1.00208
\]

This system depends on the comma being just less than a quarter of a semitone.
12. Later references, explanations and worked examples

About a year after these derivations, some standard texts on temperament confirmed the above system as a very general case of meantone. It also became clear why many musicians have difficulty with temperaments. The problem seems to be that musical texts (eg Refs 3 and 4) begin with a brief outline of many definitions, originated by subtle ancient mathematicians. Little space is given to create understanding, and no sample exercises, which would be needed even by a technical mind. These texts then move to detailed lists of tunings, their history and musical effect. At another extreme, popularizing technical texts (eg Ref 5) often assume that musicians can only add numbers or intervals, and that multiplication and logarithms, required for temperaments, are too difficult. This need not be true, especially with calculators. These texts explain simple maths in great detail, which may quickly lose any reader, and end with long lists. Many players of early music, and lute makers, resort to ready-made lists of fret positions, cents, and procedures for using meters. The following explanations and examples may be useful.

There are two systems in the literature: (1) the ancient mathematical definitions using perfect intervals as a reference, with practical differences for temperaments expressed in commas; and (2) an engineering system using modern equal temperament as the reference, with measurements in cents. The system (3) in this paper is more fundamental, needing no initial reference, but just a simple step to either of the other two systems. All this is worth explaining, with a worked example to aid fluency in any system. It should be stressed that understanding does not ensure perfect tuning, but for anyone with this practical skill, the theoretical background should be more satisfying than bare lists and procedures.

System (1) and commas

This early system for meantone is based on the definition of a syntonic comma

\[(4/3)^4 (5/4) c = 4\]

This shows that four perfect fourths within two octaves leave a third which is a syntonic comma, \(c = 81/80\), wider or sharper than perfect. Each of the practical fourths can be sharpened a small amount \(c^x\)say, to give

\[(4/3)^4 (c^x)^4 (5/4) c^y = 4\]

This reduces the sharpening of the third to \(c^y\), where \(y + 4x = 1\), or \(y = 1 - 4x\). The mathematical origin of this system is seen in the use of powers of the same base, \(c\). This allows the small changes to frequency intervals, which are multiplied, to be expressed as simple additions. The early theorists used several simple fractions for \(x\), such as

\[1/3, 2/7, 1/4, 2/9, 1/5, 1/6, 1/8, 1/11\]

giving \(y\) values of

\[-1/3, -1/7, 0, 1/9, 1/5, 1/3, 1/2, 7/11\]

The value of \(x\) required for some special cases can often be seen from a neat relation. Notable cases are 1/4 for perfect thirds, 1/5 with the same amount of tempering in fourths and thirds, and 1/3 for perfect minor thirds, since a major sixth is a fourth plus a major third with \(x + y = 0\). The other cases include the popular 1/6; and 1/11 can be shown to be a good approximation for ET. It may be useful, here, to clarify a problem of nomenclature. 'Meantone' is often used for the single, maybe original, case of perfect major thirds, with apologies for using the label in the other cases. This is unnecessary, since meantone refers to the major thirds being divided into two equal tones, which is a general property of the system. For example, in the formulae of section 2: \(C = 1\), \(D = 2/p^2\), and \(E = 4/p^4\). If necessary, any confusion could be avoided by the qualification 'quarter comma'. It is technically correct, but not always helpful, to refer to Pythagorean tuning or ET as types of meantone.

In system (1), the definitions in terms of fractions of a comma are fine for theoretical discussion, and perhaps some feel for musical effect. The differences of all notes from Pythagorean, where \(x = 0\), are sometimes given in terms of \(c\) and \(x\). However, real frequency ratios and fret positions require more work, which is not derived or given in musical texts.
System (3) and fourths, $\beta$

Calculation of a frequency ratio $r$ requires a starting value of the fourth, or $\beta$ in system (3). For our worked example, $x = 1/6$ will be most useful, and $\beta = (4/3)(c)^{1/6}$. To find $(1.0125)^{1/6}$, a general power or exponent on a calculator, or ‘log’ tables, may be available. A sufficiently accurate method is to multiply 0.0125 by 1/6, or any other $x$, and add 1. This gives $\beta = 1.33333 \times 1.00208 = 1.3361$, and this is effectively one of the values taken earlier in section 6.

Now, from section 2 one takes the formulae in turn so that $C = 1$, $F = 1.3361$, $B^b = (1.3361)^2 = 1.3361 \times 1.3361 = 1.7851$, $E^b = (1.3361)^{5/2} = 1.3361 \times 1.3361 \times 1.3361 = 2 = 1.1926$, or just $B^b \times 1.3361 + 2$, etc.

All these formulae simply multiply the previous value by 1.3361, and divide where necessary to keep between 1 and 2. The procedure is similar for the descending fourths:

$G = 2 \times 1.3361 = 1.4969$, $D = 2 \times (1.3361 \times 1.3361)$, or just $G \times 1.3361 = 1.1203$, then $A = 1.6770$, $E = 1.2551$, etc. This gives the scale listed in Table 4.

If a different value of $x$ such as 1/8 is preferred, then $\beta$ becomes $1.33333 \times 1.00156 = 1.3354$, and a new set of notes is calculated. If a value of $\beta$ or $x$ is wanted for any other condition, such as a perfect interval or ET, then it is safer to use the method in section 5 with an equation for the required number of fourths, rather than hope for a neat relation.

Next, the relative fret positions are calculated from the relation $\ell / L = (r-1)/r$, as derived in section 7. In our example, the fifth fret is the interval of a fourth above the open string. This is $F$ on the C string, or $C$ on the G string, and $r$ is just 1.3361 or $\beta$.

This gives $(r-1)/r = (1.3361-1)/1.3361 = 0.3361/1.3361 = 0.2515 = \frac{\ell}{L}$.

For a tenth fret, or $B^b$ on the C string, $(r-1)/r = (1.7851-1)/1.7851 = 0.4398$, and these values are seen in Table 6. Finally, if our string length L is 623mm say, after deducting perhaps 2mm for end corrections, the fifth fret is placed a distance $\ell$ from the nut of 0.2515 x 623 = 156.7 or 157mm. The tenth fret is placed 274mm from the nut.

System (2) and cents

Now the theory and calculation can be described for system (2) based on ET and cents. Here the frequency ratio for the equal semitones is $r = s = 2^{1/12} = 1.0595$, which was also found from $\beta^{5/4}$, as seen in section 5. A finer gradation divides each semitone into one hundred ‘cents’, each with a frequency ratio $r = (2^{1/12})^{1/100} = 2^{1/1200}$, or about 1.00059. All these tiny intervals are equal ratios, so that larger intervals when expressed in cents can be added. This is the advantage of system (2), which has led to its use for precise measurements by electronic tuning meters, with frequencies measured in Hz or cycles/sec. This system also became a standard because ET had become almost universal in western music. However, no other historically developed temperament has a simple relation to ET. The only solution is to convert these more natural frequency ratios into cents. If a frequency ratio $r$ is converted to say $n$ cents then

$$r = (2^{1/1200})^n = 2^{n/1200}$$

Here $n/1200$ is just the logarithm of $r$ using a base of 2. For using normal base 10 logarithms

$$n = (1200 / \log_{10} 2) \log_{10} r = 3987 \log_{10} r$$

None of the perfect intervals retains a simple form. For example, a perfect fifth has a value of $r$ equal to 3/2 or 1.5, which in cents is $n = 3987(\log 1.500) = 0.1761 \times 3987 = 702$ cents, compared with 700cents for a fifth in ET. Similarly, a perfect fourth is 498cents, showing the additive property. A perfect third is 386cents, which shows the large sharpness of 400cents in ET. Using many additions of cents to illustrate commas from cycles of perfect fifths may seem perverse. However, twelve perfect fifths are about 24 cents sharp, or the Pythagorean comma in cents. Also, four perfect fourths or fifths give a third that is about 8cents wider than ET, and therefore about 22cents wider than perfect, or the syntonic comma. In general, 1cent accuracy is not sufficient for repeated additions, and checks with the relation between $n$ and $r$ are useful. For our
example of sixth comma meantone, a fifth has $r$ at 1.497 giving $n$ as about 698½ cents, and for a major third $r$ is 1.255 or 394 cents.

Deviations of the fifths from ET for $x = 0, 1/6$ and $1/4$ are $+2$, $-1\frac{1}{2}$, and $-3\frac{1}{2}$ cents. All the notes of a scale can be found by summing the deviations for successive fourths or fifths. The tuning of these three cases relative to ET for the previous lute scale can be given in nutshell as:

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D♭</th>
<th>D</th>
<th>E♭</th>
<th>E</th>
<th>F</th>
<th>G♭</th>
<th>G</th>
<th>A♭</th>
<th>A</th>
<th>B♭</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 0$ Pythagorean</td>
<td>0</td>
<td>-10</td>
<td>+4</td>
<td>-6</td>
<td>+8</td>
<td>-2</td>
<td>-12</td>
<td>+2</td>
<td>-8</td>
<td>+6</td>
<td>-4</td>
<td>+10</td>
</tr>
<tr>
<td>$x = 1/11$ ET</td>
<td>0</td>
<td>+8</td>
<td>-3</td>
<td>+5</td>
<td>-6</td>
<td>+1\frac{1}{2}</td>
<td>+10</td>
<td>-1\frac{1}{2}</td>
<td>+6</td>
<td>-5</td>
<td>+3</td>
<td>-8</td>
</tr>
<tr>
<td>$x = 1/4$ comma</td>
<td>0</td>
<td>+18</td>
<td>-7</td>
<td>+11</td>
<td>-14</td>
<td>+3\frac{1}{2}</td>
<td>+21</td>
<td>-3\frac{1}{2}</td>
<td>+14</td>
<td>-11</td>
<td>+7</td>
<td>-18</td>
</tr>
</tbody>
</table>

This shows very clearly how interpolation in cents can be used between two extremes, as in the initial main tables and diagram. Once the essentials have been grasped, long lists are unnecessary. In any system, one could also give deviations from $x = 0$, and tune from perfect fifths using calculated rates of beating, starting with a tuning fork.

Later, I also found a very short, non-mathematical, but excellent, treatment of temperaments for keyboards in terms of the Pythagorean comma (Ref 6). For keyboards it is possible and desirable to have slightly different adjustments to successive fifths, in order to return to the initial pitch, say C, and allow dual use of a single enharmonic, such as G♯/A♭. There was a modification of meantone to achieve this. In the simplest conceivable temperament, the whole comma could be taken off a single fifth. A baroque system, Kirnberger 2, narrowed just two fifths by half a comma each. Meantone can be described to a good approximation by the Pythagorean comma, since this is only about 10% greater than the syntonic comma. This gives ET exactly as ‘twelfth Pythagorean comma’. A popular practical system narrowed each of six successive fifths by 1/6th of a Pythagorean comma, and left the others perfect. This is Valotti’s temperament, which may be preset on tuning meters, and therefore provides half the notes needed for sixth comma meantone, but not those with greater deviations that can be useful for tuning. It is sometimes heard that it is preferable to bias the 2nd and 3rd courses slightly flat, and the 4th and 5th sharp, as this would tend to be nearer to ‘meantone’ than ET. This is clearly true for the six open notes, but not for all the fretted notes. It can be shown from the deviations above that all sorts of bias have a similar inaccuracy. This is greater than just tuning as well as possible to the temperament set by the frets.

References

**EFFECTS OF TEMPERAMENTS ON CONSORTS**

Two years after this first main paper, during a talk on different sizes of lute it was asked what would happen if lutes of different pitches, and tuned in temperaments other than ET, tried to play together. In other words, how do temperaments and different lute sizes affect each other? This presented an interesting and practical problem, although temperaments had not been a continual preoccupation. The following analysis formed a letter to Lute News 78.

This solution depends on the principle that when lutes in different pitches or keys are playing together, the alternative sharps and flats should ideally be the same on all lutes. This was not an obvious starting point, but having seen it the relations in the earlier paper became useful.
For example, taking a lute in G as a reference, the top string would most likely have

\[G, A^b, A, B, C, D^b, D, E^b, E, F^a, G\]

where \(A^b\) has been chosen instead of the different pitch \(G^a\), etc. These notes should then be set on the top string of the other instruments playing with the G lute. The change in these notes across the various strings will also be the same for all the lutes, which is a key point of the solution. This results in geometrically different fret positions on lutes at different pitches.

As a likely example, it follows that a D lute with frets initially in the same position as the G lute, should have the 11th fret moved from a sharp to a flat, for changing a C\(^\#\) into a D\(_b\). Similarly, for an A lute, move frets 4 and 11; for an E lute move frets 4, 9 and 11; and for an F\(^a\) lute move frets 2, 4, 7, 9 and 11. Then, for a C lute move the sixth fret from a flat to a sharp, and similarly for an F lute move frets 1 and 6. The detailed reasons for these changes, if there is any doubt, can be seen from tracing notes on the various strings. This can use musical theory, or the note formulae in section 3 of the paper.

All this indicates that an ‘unaltered’ D lute and G lute duet, and others a fourth or fifth apart, would have only slight tuning problems. However, there are more difficulties for F and G duets, and others a tone, or two fifths apart. Notice that the change needed for a D lute is less than for a C lute, and similarly for A and F lutes. There would be even more problems for lutes a major or minor third, or even a semitone apart. The greatest effect would occur for six fifths, or two minor thirds. For some combinations of lutes, it would be useful to choose a different reference lute. In a D, E and A lute trio, a better reference could be an A lute. Using a lower pitched E, F or F\(^a\) lute as a more sonorous solo G lute could produce a practical dilemma for the further use of playing in a consort.

These tuning effects stem from the need to specify temperaments consistently. These effects are additional to the usual problems of different fingering patterns needed for differently pitched instruments. (For this reason, duets for clarinets in A and B\(_b\), or worse still E\(_b\), would be difficult. These pitches or sizes were devised to play in various timbres and keys with other instruments, and not as a family consort like some other types of instrument.) Music for lute consorts tends to be centred on a few common keys, which would minimize these problems. It may be interesting to examine the music for instances where ‘unaltered’ lutes would have a tuning problem. The music might sometimes take advantage of differences, such as good F\(^a\) on a D lute where a G lute may have a problem.

If these fundamental points are found to be musically significant, the practical changes in fret position can be found from the tables and diagram in the main paper. However, a neat general relation for a required change in position can be derived from the formulae. For any enharmonic pair, the ratio of the frequency of any flat to its sharp twin is seen to be \(\beta^{12/32}\), from Tables 2 or 3. For a general ‘x comma meantone’ the ratio of sounding lengths between any sharp and its flat twin is therefore \((4/3)(81/80)^x\)^{12/32}. This can be calculated for any value of x, and for the popular sixth comma meantone, the ratio of lengths is 1.0114. From this, the change in fret position for a lute with a string length of 600mm would vary between slightly less than 6.8mm for fret 1, to slightly more than 3.4mm for fret 11. As an exact example using Table 7, in order to change the sixth fret to a sharp in SCM, the fret needs to be moved nearer to the nut by 4.8mm. This follows from 0.0114(600 - 178.1)mm.

For equal temperament, \(\beta^{12/32} = 1\), and there is no problem of alternative fret positions, as expected. This would greatly simplify different sized lutes playing together, or with other instruments not having flexible tuning. Historically, this may have been a practical pressure to use ET on many lutes. Lutes for solo music, or with flexible instruments and voices, could more easily use an unequal temperament. Some sets of pieces in several keys or modes may have been intended to exploit tuning differences when using unequal systems. The use of ET in such pieces would be mainly a matter of differences in technique, available note patterns and range, apart from any small departure from an exact ET, which was mentioned in section 5.
<table>
<thead>
<tr>
<th>Note</th>
<th>Perfect scale frequency ratios</th>
<th>Attempt to show symmetry of intervals</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>1</td>
<td>1.0000</td>
</tr>
<tr>
<td>C#</td>
<td>135/128</td>
<td>1.0547</td>
</tr>
<tr>
<td>D</td>
<td>16/15</td>
<td>1.0667</td>
</tr>
<tr>
<td>D♭</td>
<td>9/8</td>
<td>1.1250</td>
</tr>
<tr>
<td>D♯</td>
<td>75/64</td>
<td>1.1719</td>
</tr>
<tr>
<td>E♭</td>
<td>6/5</td>
<td>1.2000</td>
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<td>E</td>
<td>5/4</td>
<td>1.2500</td>
</tr>
<tr>
<td>F</td>
<td>4/3</td>
<td>1.3333</td>
</tr>
<tr>
<td>F♯</td>
<td>45/32</td>
<td>1.4062</td>
</tr>
<tr>
<td>G♭</td>
<td>64/45</td>
<td>1.4222</td>
</tr>
<tr>
<td>G</td>
<td>3/2</td>
<td>1.5000</td>
</tr>
<tr>
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<td>25/16</td>
<td>1.5625</td>
</tr>
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<td>8/5</td>
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<td>225/128</td>
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<td>1.7778</td>
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<td>B</td>
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</tr>
<tr>
<td>C</td>
<td>2</td>
<td>2.0000</td>
</tr>
</tbody>
</table>

Note

- 135/128 greater than 16/15
- 135/128 greater than 25/24
- 25/24 smaller than 16/15
- Chromatic semitones diatonic semitones
- 16/15 diatonic semitone
- 9/8 major tone
- 10/9 minor tone
- 9/8
- 10/9
- 9/8
- 16/15
### Table 2
Frequency ratio formulae for a general temperament

<table>
<thead>
<tr>
<th>Note</th>
<th>Frequency ratio</th>
<th>Further enharmonics</th>
</tr>
</thead>
<tbody>
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</tr>
<tr>
<td>C#</td>
<td>8/β^7</td>
<td></td>
</tr>
<tr>
<td>D♭</td>
<td>β^5/4</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>2/β^2</td>
<td>C## 64/β^14, Ebb β^10/16</td>
</tr>
<tr>
<td>D#</td>
<td>16/β^9</td>
<td></td>
</tr>
<tr>
<td>E♭</td>
<td>β^3/2</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>4/β^4</td>
<td>F♭ β^5/8</td>
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<tr>
<td>F</td>
<td>β</td>
<td>E# 32/β^11, Gbb β^11/32</td>
</tr>
<tr>
<td>F#</td>
<td>8/β^6</td>
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<td>β^6/4</td>
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<tr>
<td>G</td>
<td>2/β</td>
<td>F## 64/β^13, A♭♭ β^11/16</td>
</tr>
<tr>
<td>G#</td>
<td>16/β^8</td>
<td></td>
</tr>
<tr>
<td>A♭</td>
<td>β^4/2</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>4/β^3</td>
<td>G## 128/β^15, B♭♭ β^9/8</td>
</tr>
<tr>
<td>A#</td>
<td>32/β^10</td>
<td></td>
</tr>
<tr>
<td>B♭</td>
<td>β^2</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>8/β^5</td>
<td>C♭ β^7/8</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>B♭# 32/β^12</td>
</tr>
</tbody>
</table>

### Table 3
Frequency ratios for a selected lute scale

<table>
<thead>
<tr>
<th>Note</th>
<th>Frequency ratio</th>
<th>Semitone interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>1</td>
<td>β^5/4</td>
</tr>
<tr>
<td>D♭</td>
<td>β^5/4</td>
<td>8/β^7</td>
</tr>
<tr>
<td>D</td>
<td>2/β^2</td>
<td>β^5/4</td>
</tr>
<tr>
<td>E♭</td>
<td>β^3/2</td>
<td>8/β^7</td>
</tr>
<tr>
<td>E</td>
<td>4/β^4</td>
<td>β^5/4</td>
</tr>
<tr>
<td>F</td>
<td>β</td>
<td>β^5/4</td>
</tr>
<tr>
<td>G♭</td>
<td>β^6/4</td>
<td>8/β^7</td>
</tr>
<tr>
<td>G</td>
<td>2/β</td>
<td>β^5/4</td>
</tr>
<tr>
<td>A♭</td>
<td>β^4/2</td>
<td>8/β^7</td>
</tr>
<tr>
<td>A</td>
<td>4/β^3</td>
<td>β^5/4</td>
</tr>
<tr>
<td>B♭</td>
<td>β^2</td>
<td>8/β^7</td>
</tr>
<tr>
<td>B</td>
<td>8/β^5</td>
<td>β^5/4</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>β^5/4</td>
</tr>
</tbody>
</table>
### Table 4

Frequency ratios for various lute temperaments

<table>
<thead>
<tr>
<th>Value of fourth, $\beta$</th>
<th>Low limit of fourth</th>
<th>Perfect fourth</th>
<th>Equal temperament</th>
<th>SCM</th>
<th>Perfect major third</th>
<th>Perfect minor third</th>
<th>High limit of fourth</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.3300</td>
<td>1.3333</td>
<td>1.3348</td>
<td>1.3360</td>
<td></td>
<td>1.3375</td>
<td>1.3390</td>
<td>1.3400</td>
</tr>
<tr>
<td>C</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>Db</td>
<td>1.040</td>
<td>1.053</td>
<td>1.059</td>
<td>1.064</td>
<td>1.070</td>
<td>1.076</td>
<td>1.080</td>
</tr>
<tr>
<td>D</td>
<td>1.130</td>
<td>1.125</td>
<td>1.122</td>
<td>1.120</td>
<td>1.118</td>
<td>1.116</td>
<td>1.114</td>
</tr>
<tr>
<td>Eb</td>
<td>1.176</td>
<td>1.185</td>
<td>1.189</td>
<td>1.192</td>
<td>1.196</td>
<td>1.200</td>
<td>1.203</td>
</tr>
<tr>
<td>E</td>
<td>1.278</td>
<td>1.266</td>
<td>1.260</td>
<td>1.255</td>
<td>1.250</td>
<td>1.244</td>
<td>1.241</td>
</tr>
<tr>
<td>Gb</td>
<td>1.384</td>
<td>1.405</td>
<td>1.414</td>
<td>1.422</td>
<td>1.431</td>
<td>1.441</td>
<td>1.447</td>
</tr>
<tr>
<td>G</td>
<td>1.504</td>
<td>1.500</td>
<td>1.498</td>
<td>1.497</td>
<td>1.495</td>
<td>1.494</td>
<td>1.492</td>
</tr>
<tr>
<td>Ab</td>
<td>1.564</td>
<td>1.580</td>
<td>1.587</td>
<td>1.593</td>
<td>1.600</td>
<td>1.607</td>
<td>1.612</td>
</tr>
<tr>
<td>A</td>
<td>1.700</td>
<td>1.687</td>
<td>1.682</td>
<td>1.677</td>
<td>1.672</td>
<td>1.667</td>
<td>1.662</td>
</tr>
<tr>
<td>Bb</td>
<td>1.769</td>
<td>1.778</td>
<td>1.782</td>
<td>1.785</td>
<td>1.789</td>
<td>1.793</td>
<td>1.796</td>
</tr>
<tr>
<td>B</td>
<td>1.922</td>
<td>1.898</td>
<td>1.888</td>
<td>1.880</td>
<td>1.869</td>
<td>1.859</td>
<td>1.852</td>
</tr>
<tr>
<td>C</td>
<td>2.000</td>
<td>2.000</td>
<td>2.000</td>
<td>2.000</td>
<td>2.000</td>
<td>2.000</td>
<td>2.000</td>
</tr>
</tbody>
</table>

Underline indicates a perfect interval
Table 5

Frequency ratios for limiting values of β and averages

<table>
<thead>
<tr>
<th>Note</th>
<th>Low limit ( β = 1.330 )</th>
<th>High limit ( β = 1.340 )</th>
<th>Average ( β = 1.335 )</th>
<th>Average of high and low limits</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>C#</td>
<td>1.0867</td>
<td>1.0312</td>
<td>1.0586</td>
<td>1.0589</td>
</tr>
<tr>
<td>D\b</td>
<td>1.0404</td>
<td>1.0801</td>
<td>1.0601</td>
<td>1.0603</td>
</tr>
<tr>
<td>D</td>
<td>1.1306</td>
<td>1.1138</td>
<td>1.1222</td>
<td>1.1222</td>
</tr>
<tr>
<td>D#</td>
<td>1.2287</td>
<td>1.1486</td>
<td>1.1879</td>
<td>1.1887</td>
</tr>
<tr>
<td>E\b</td>
<td>1.1763</td>
<td>1.2030</td>
<td>1.1897</td>
<td>1.1897</td>
</tr>
<tr>
<td>E</td>
<td>1.2784</td>
<td>1.2406</td>
<td>1.2593</td>
<td>1.2595</td>
</tr>
<tr>
<td>E#</td>
<td>1.3893</td>
<td>1.2794</td>
<td>1.3331</td>
<td>1.3343</td>
</tr>
<tr>
<td>F\b</td>
<td>1.2238</td>
<td>1.2994</td>
<td>1.2611</td>
<td>1.2616</td>
</tr>
<tr>
<td>F</td>
<td>1.3300</td>
<td>1.3400</td>
<td>1.3350</td>
<td>1.3350</td>
</tr>
<tr>
<td>F#</td>
<td>1.4454</td>
<td>1.3819</td>
<td>1.4132</td>
<td>1.4136</td>
</tr>
<tr>
<td>G\b</td>
<td>1.3837</td>
<td>1.4473</td>
<td>1.4152</td>
<td>1.4155</td>
</tr>
<tr>
<td>G</td>
<td>1.5037</td>
<td>1.4925</td>
<td>1.4981</td>
<td>1.4981</td>
</tr>
<tr>
<td>G#</td>
<td>1.6342</td>
<td>1.5392</td>
<td>1.5859</td>
<td>1.5867</td>
</tr>
<tr>
<td>A\b</td>
<td>1.5645</td>
<td>1.6121</td>
<td>1.5881</td>
<td>1.5883</td>
</tr>
<tr>
<td>A</td>
<td>1.7002</td>
<td>1.6624</td>
<td>1.6812</td>
<td>1.6813</td>
</tr>
<tr>
<td>A#</td>
<td>1.8477</td>
<td>1.7144</td>
<td>1.7797</td>
<td>1.7810</td>
</tr>
<tr>
<td>B\b</td>
<td>1.7689</td>
<td>1.7956</td>
<td>1.7822</td>
<td>1.7822</td>
</tr>
<tr>
<td>B</td>
<td>1.9223</td>
<td>1.8517</td>
<td>1.8866</td>
<td>1.8870</td>
</tr>
<tr>
<td>B#</td>
<td>2.0891</td>
<td>1.9095</td>
<td>1.9971</td>
<td>1.9993</td>
</tr>
<tr>
<td>C\b</td>
<td>1.8403</td>
<td>1.9394</td>
<td>1.8893</td>
<td>1.8898</td>
</tr>
<tr>
<td>C</td>
<td>2.0000</td>
<td>2.0000</td>
<td>2.0000</td>
<td>2.0000</td>
</tr>
</tbody>
</table>
Table 6

Relative fret positions from the nut, for temperaments in Table 4

<table>
<thead>
<tr>
<th>Fret number</th>
<th>Perfect fourth ( \beta = 1.3333 )</th>
<th>Equal temperament 1.3348</th>
<th>SCM 1.3360</th>
<th>Perfect major third 1.3375</th>
<th>Perfect minor third 1.3390</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st or b</td>
<td>0.0503</td>
<td>0.0557</td>
<td>0.0601</td>
<td>0.0654</td>
<td>0.0706</td>
</tr>
<tr>
<td>2nd c</td>
<td>0.1111</td>
<td>0.1087</td>
<td>0.1071</td>
<td>0.1055</td>
<td>0.1039</td>
</tr>
<tr>
<td>3rd d</td>
<td>0.1561</td>
<td>0.1590</td>
<td>0.1611</td>
<td>0.1639</td>
<td>0.1667</td>
</tr>
<tr>
<td>4th e</td>
<td>0.2088</td>
<td>0.2063</td>
<td>0.2032</td>
<td>0.2000</td>
<td>0.1961</td>
</tr>
<tr>
<td>5th f</td>
<td>0.2500</td>
<td>0.2509</td>
<td>0.2515</td>
<td>0.2523</td>
<td>0.2532</td>
</tr>
<tr>
<td>6th g</td>
<td>0.2882</td>
<td>0.2928</td>
<td>0.2968</td>
<td>0.3012</td>
<td>0.3060</td>
</tr>
<tr>
<td>7th h</td>
<td>0.3333</td>
<td>0.3324</td>
<td>0.3320</td>
<td>0.3311</td>
<td>0.3307</td>
</tr>
<tr>
<td>8th i</td>
<td>0.3670</td>
<td>0.3699</td>
<td>0.3722</td>
<td>0.3750</td>
<td>0.3777</td>
</tr>
<tr>
<td>9th k</td>
<td>0.4072</td>
<td>0.4055</td>
<td>0.4037</td>
<td>0.4019</td>
<td>0.4000</td>
</tr>
<tr>
<td>10th l</td>
<td>0.4376</td>
<td>0.4388</td>
<td>0.4398</td>
<td>0.4410</td>
<td>0.4423</td>
</tr>
<tr>
<td>11th m</td>
<td>0.4732</td>
<td>0.4703</td>
<td>0.4680</td>
<td>0.4649</td>
<td>0.4621</td>
</tr>
<tr>
<td>12th n</td>
<td>0.5000</td>
<td>0.5000</td>
<td>0.5000</td>
<td>0.5000</td>
<td>0.5000</td>
</tr>
</tbody>
</table>

Table 7

Fret positions (mm) for a 600mm string length, from Tables 6 and 4

<table>
<thead>
<tr>
<th>Fret number</th>
<th>Perfect fourth ( \beta = 1.3333 )</th>
<th>Equal temperament 1.3348</th>
<th>SCM 1.3360</th>
<th>Perfect major third 1.3375</th>
<th>Perfect minor third 1.3390</th>
<th>Variation (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st or D</td>
<td>30.2</td>
<td>33.4</td>
<td>36.1</td>
<td>39.2</td>
<td>42.4</td>
<td>+12</td>
</tr>
<tr>
<td>2nd D</td>
<td>66.7</td>
<td>65.2</td>
<td>64.3</td>
<td>63.3</td>
<td>62.4</td>
<td>-4</td>
</tr>
<tr>
<td>3rd E</td>
<td>93.7</td>
<td>95.4</td>
<td>96.6</td>
<td>98.3</td>
<td>100.0</td>
<td>+6</td>
</tr>
<tr>
<td>4th E</td>
<td>125.3</td>
<td>123.8</td>
<td>121.9</td>
<td>120.0</td>
<td>117.7</td>
<td>-8</td>
</tr>
<tr>
<td>5th F</td>
<td>150.0</td>
<td>150.5</td>
<td>150.9</td>
<td>151.4</td>
<td>151.9</td>
<td>+2</td>
</tr>
<tr>
<td>6th G</td>
<td>172.9</td>
<td>175.7</td>
<td>178.1</td>
<td>180.7</td>
<td>183.6</td>
<td>+11</td>
</tr>
<tr>
<td>7th G</td>
<td>200.0</td>
<td>199.5</td>
<td>199.2</td>
<td>198.7</td>
<td>198.4</td>
<td>-2</td>
</tr>
<tr>
<td>8th A</td>
<td>220.2</td>
<td>221.9</td>
<td>223.3</td>
<td>225.0</td>
<td>226.6</td>
<td>+6</td>
</tr>
<tr>
<td>9th A</td>
<td>244.3</td>
<td>243.4</td>
<td>242.2</td>
<td>241.1</td>
<td>240.0</td>
<td>-4</td>
</tr>
<tr>
<td>10th B</td>
<td>262.6</td>
<td>263.3</td>
<td>263.9</td>
<td>264.6</td>
<td>265.4</td>
<td>+3</td>
</tr>
<tr>
<td>11th B</td>
<td>283.9</td>
<td>282.2</td>
<td>280.8</td>
<td>279.0</td>
<td>277.2</td>
<td>-7</td>
</tr>
<tr>
<td>12th C</td>
<td>300.0</td>
<td>300.0</td>
<td>300.0</td>
<td>300.0</td>
<td>300.0</td>
<td>-3</td>
</tr>
</tbody>
</table>
Fret positions for various temperaments

\[ \beta = 1.3333 \]

\[ \beta = 1.3348 \]

\[ \beta = 1.3360 \]

\[ \beta = 1.3375 \]

\[ 1.3390 \]
ELASTICITY OF LUTE STRINGS (2)

This communication contains four parts: (1) an initial short letter, from Lute News 78, on a criterion for the elastic modulus of lute strings, followed by (2) a larger paper on the elastic properties which would be needed to explain the angled bridges of old lutes. This paper has additions on modern metal-wound strings, and the initial idea of an entirely new explanation for angled bridges. Part (3) provides mathematical details of the elastic theory, and (4) is a short letter, from Lute News 81, suggesting a new modern string material. Further background is in the general introduction to paper 1.

1. AN ELASTIC MODULUS CRITERION

Lute players will have noticed that their lower pitched strings in particular sound sharper when plucked strongly. A tuning meter or good ear detects as much as 20 cents or a fifth of a semitone in pitch variation. The sharpness disappears as the vibration amplitude decays, but there is often a problem in obtaining good tuning, and a related dull tone. Also, some reviews of tuning meters in Lute News 74 remarked that this pitch variation can cause practical difficulty.

The reason for this effect is that the increased stretching, and hence tension, caused by plucking, is comparable with the stretching required for initial tuning. For the higher strings, the relative effect is much smaller and there is negligible sharpening and a good tone. Players will have found that when fitting new strings, the higher ones need many turns to reach pitch, whereas little peg action is needed for lower strings. I have been asked to explain this point, made thirty years ago (eg Ref 1) and indeed three centuries ago (Ref 2), since it may not be well known to all devotees of the lute.

I have found that the basic relations for vibrating strings can be developed to predict what stiffness in stretching, or elastic modulus, is needed for large amounts of initial stretching when tuning all the lower strings. The elastic modulus of a string should ideally be proportional to the square of its frequency. This criterion is for strings of equal density and sounding length. Very interestingly, it is independent of tension and diameter, which become subsequent related choices.

This overall theoretical guide to changing the effective modulus may help string makers in devising a variety of twists, braids, ropes etc, in nylon, gut and other materials. For wound or loaded strings, or mixtures of string types, there are many options and the criterion requires the effective modulus divided by the effective density. However, the opening remark on plucking refers to some metal-wound strings as well as thicker monofilaments, which indicates that current wound strings still have somewhat high modulus or low proportions of added mass. A practical approach may start some twisting etc, including threads of denser material and metal, at the third or fourth course, and progressively increase density for the lower strings. This could use synthetic materials for consistency and low cost, unless natural ones are so much better, or necessary.

The predicted need for a decreased elastic modulus can be shown to reduce several other non-linear complications, in addition to the initial amplitude dependent frequency. It can also reduce progressive sharpening of higher overtones caused by flexure. The bending force increases with string diameter, and together with non-linearities and friction, causes poor tone and rapid decay with thicker monofilaments. In contrast, modern precision metal-wound strings with a thin stranded core have little damping, and may be less desirable than the alternative gut.
Non-uniformities in natural materials can be expected to produce a random inharmonicity, but also effective damping, and similar effects may be available in synthetics. Twisting etc could also provide internal friction. However, it is difficult to pinpoint these complexities when there may be no surviving old strings, but there may be useful acoustic parallels with the irregularities of old handmade wind instruments.

2. STRINGS AND ANGLED BRIDGES ON OLD LUTES

This paper is intended to be concise, drawing several detailed aspects of strings into a short narrative with no explicit maths. Background theory and equations are given in part (3).

1. The Problem
Many surviving old lutes have bridges that are angled at about 3°, so that the treble strings are about 3mm longer than the sixth course bass strings (Ref 3). These appear to be in an original playable form, and not discarded mistakes or later alterations. For whatever reason the bridges were angled, they would still need to be compatible with good tuning.

Good tuning for both treble and bass strings would require suitable strings, frets and action height. The maximum degree of flatness for such trebles is of order 603/600 = 1.005, which is 1/12th of a semitone, or 8 cents. This is a significant deviation, but not enormous. However, some trebles are longer by about 6mm, which doubles the effect.

If simple non-angled frets were set up they would need to provide truer tuning, in a chosen temperament, for the bass strings than for the trebles. This is because strings cannot be flattened during playing, and the treble may only be sharpened by an increased tension when it is pressed down to a fret and stretched slightly. These stretches are much smaller, 0.01 to 0.1mm, than the relative changes of fret position, 1mm. An average fret position would not be useful, and considerably angled frets are unstable, give less scope for other fine adjustments, are inelegant and generally disliked by makers. However, any proof of such frets could invalidate later conclusions, and so would an angled nut or a spacer at the bridge.

N.B. It is appropriate to add here, with two years hindsight, that this early attempt assumed implicitly that strings have a uniform diameter or distribution of mass, which is the familiar modern situation. The initial positive reasoning in this early work will be retained throughout this paper, but it can be viewed later as a necessary ‘reductio contradiction’ against string elasticity as a main explanation for angled bridges. There is still some useful understanding of several aspects of elasticity, including sharpening by frets.

2. String elasticity
Over recent decades there has been considerable debate, and practical success, on reducing the tensile stiffness of bass strings. Thinking theoretically, I considered recently that a useful practical goal might be an equal large stretch during the initial tuning for all strings. This predicts that the elastic modulus (divided by density, for different materials) needs to be proportional to the frequency squared. This criterion indicates an enormous lowering of modulus (about 1/16th) from a first to a sixth course, two octaves lower. These ideas are explained in greater detail in the other parts of this communication. Practical advantages are reductions in: sharpening for fretted notes; transient sharpening when plucking; problems of damping and tone; and sharpening of higher overtones caused by bending of a string. This goal would be useful for many modern strings, which still have similar effective moduli for treble, middle and bass strings.
Turning to the problem of old strings, my initial thought was that even for a non-angled bridge the elastic modulus criterion already ideally requires a large difference between trebles and basses. Therefore, an angled bridge would need even lower stiffness or modulus in the basses, and/or greater stiffness in the treble. Previously, rather smaller degrees of elasticity in the bass have been considered qualitatively to be sufficient, by an analogy with modern guitars, which have longer basses to compensate for their stiffness. Alternatively, a stiffer treble string was suggested in Ref 3. The present initial condition to be explored rigorously is a very strong combined effect. As a possible bonus, it has often been noted that an increased stiffness with gut treble strings can give extra feel in legato playing. A further advantage could be a greater stability of tuning in the bass, with a greater action height for the various types of stretching effect listed above.

Initially, I wanted to avoid detailed calculations, yet still felt a need for some practical numbers. These opened up the problem and led to some resolution, but only temporary in this paper.

3. Outline of calculations

As indicated in section 1, a bass string with no sharpening effect, which would require a very low modulus, provides a suitable theoretical reference point. Firstly, accurate fret positions for equal temperament were calculated. Then I calculated the degrees of flattening, in terms of frequency ratios, that these fret positions would give for a slightly longer treble string having no sharpening effect. Next, I calculated the small amounts of stretching when the treble string is pressed down to a fret. Finally, the elastic properties of a treble were combined with the extra stretching to calculate frequency increases, and then compare these with the increases that are needed to compensate for the slightly longer treble sounding lengths.

As a purely theoretical exercise, for each fret one can calculate paired values of stress, which is related to frequency, and strain, which is the relative total extension. However, this can only give a limited form of stress-strain function, and much detailed testing might be needed to assess its practicality, or find a suitable material. Here, for practical illustration I used a shortcut with an experimental curve for a nylon treble f' string with a diameter of 0.50mm. This required a very long test, equivalent to several settling periods when fitting a new string. In addition to the usual loading, a similar unloading curve was necessary. Informative tests were also made on middle f strings in both nylon and PVF, and a low F metal-wound Pyramid. The octave intervals enable useful comparisons.

During this work, in addition to the stretch produced for a string to just meet a fret, the extra stretch from pushing down a string between different pairs of frets became important. Some of these stretching effects had been estimated at the end of paper 1. The next sections give detailed results and conclusions.

4. Fret positions and thickness

(i) The degree of flattening, for the 3mm longer treble with no stretch sharpening, was found to be 1.0003 or ½ cent at fret 1 or b, near the nut, increasing to 1.0017 or 3 cents at fret 5 or f, then to 1.0050 or 8 cents at fret 12 or n.

(ii) Typical modern fret diameters, 0.9mm for fret b and decreasing to 0.5 mm for fret i, were used for a start. The calculations gave a similar stretch, of about 0.05mm, for fully pressing down at every fret. This would produce a similar factor of sharpening for all frets. This explains why theoretical fret positions are a good start, and why modern players may not be aware of fret sharpening, until perhaps open strings are tuned against high frets on much lower strings with a high action. These calculations require much geometry with a reasonable model, and results by drawing would not be easy. This constant amount of stretch was not useful for an explanation of angled bridges, which requires larger stretches and frequency increases at higher frets. Its effect would be similar to an average fret position in (i). In detail, the total stretch of 0.05mm is a
combination of a large effect at fret b, caused by the large fret diameter, which decreases towards n; and an opposite increasing effect due to greater action heights from b to n.

(iii) A second calculation used a large diameter of 1.0mm for all the frets. The large stretch at b due to fret diameter continued towards n, and also increased strongly as the thicker, higher frets became closer together. The total stretch increased from about 0.05mm at b to 0.12mm at n.

(iv) It became clear that to obtain a stretch that increased gradually from a low amount at b to a moderate value at n, required a fret thickness that was smaller at b and increased towards n. With this ‘reversed fretting’, a diameter of 0.4mm was used for b, increasing to 1.1mm at fret i, and this produced a steady increase of stretch. There was considerable sensitivity to the string height near the nut, which was also set at 0.4mm. The clearances between a fretted string and the other frets were rather small, and ranged from 0.1 to 0.6mm. This was for a string height at the bridge of 4.0mm. This could be increased, or the neck sloped forward very slightly, and the precise effect at b relaxed a little. On a modern lute these tight clearances only apply to the fret just above the one in use.

N.B. An initial adjustment of reverse frets would be difficult, and their use may not be practical, plausible, or historical. Here they are part of exploring how elasticity could explain angled bridges, and may be viewed later as part of a ‘reductio’.

(v) The amounts of stretch given by this reverse fretting, when combined with the stress function for nylon, remarkably gave just the required degree of sharpening for all the frets from b to i on the treble string. Any higher frets are needed only for the upper strings, and their positions would not depend on lower strings. This work showed encouragingly that quantitative analysis of fret sharpening was possible. There is an assumption in these calculations that the extra stretch is distributed throughout a whole string. This is reasonable for normal playing, but some effects may dig a string into a fret. A large force is needed for this, as with pressing very close to a bridge or nut, because the stretch becomes comparable with the displacement near these ends. Rough or metal-wound strings might lock onto a fret.

5. Perfect and practical bass strings

The above system of reverse fretting holds for the perfect bass string with no sharpening effect, which would need a very low modulus to give a large initial stretch on tuning an open string. This is why a simple nylon treble, which would normally be considered more elastic than a practical bass, showed significant relative fret sharpening.

A further useful step is to imagine a deeper perfect bass, with a longer treble string which needs as much as 1¼ or even 1½ times the stiffness of the nylon. The real, or intermediate, bass can now be shown to need a non-zero stiffness ¼ or ½ of the initial treble, or nylon, which means 1/5th or 1/3rd of the new treble value. In this virtual reference system neither bass nor treble strings would use ideal theoretical fret positions. Although an increased treble stiffness could give extra feel in playing, an acceptable limit would soon be reached. This scheme indicates some relief from the strict modulus criterion, but the need for a low bass modulus cannot be removed entirely, and it must always be considerably lower than the treble value. These stiffer strings might also reduce the need for the large increases of fret diameter. In the design of a real lute the four middle strings would require intermediate properties.

N.B. The need for a complex grading of extreme properties could also be included in the ‘reductio’. The following section has some comments that appeared relevant to this strange, but at the time inescapable, explanation for angled bridges. With hindsight, much of the practical conjecture is wrong, but the science became useful.
6. Discussion
A mechanism of reverse fretting with a strongly decreasing elastic modulus for lower strings
would be expected to give completely different playing and listening experiences from modern
lutes. The dependency on modulus suggests a very precise and subtle system for the old makers
and players. Natural gut is the most probable material for old strings, with a later addition of a
metal winding. Other possibilities are silk, horsehair, flax, and metal salts. Another old
candidate may be a fine decorative thread made from very fine metal spun onto a thin natural
fibre. The only surviving tradition is in Lapland, according to the V&A Museum.
Shortly after this work, a discussion of Capriola’s instructions (Ref 4) seemed relevant. The
passage is: ‘And know what fretting the lute is about: do so that the first fret nearly touches the
strings, and so forth to the end, since the closer the fret is to the string, the strings so to speak
‘arpiza’ and the lute seems to speak better’. It is then noted that ‘arpiza’ may mean a harp with a
clear sound, or perhaps a bray harp with buzzing notes. The phrase ‘and so forth to the end’
might indicate reversed fretting which produced buzzing, or simply the normal small clearance
for standard frets. There may be some evidence for reversed frets in Holbein’s ‘The
Ambassadors’ painted in 1533, even though it is famous for the rod-like skull with an extreme
viewpoint. The fret thickness appears to increase from b to i, perhaps a doubling, comparable
with the third and fifth course strings. All the strings have a buff colour and a well-defined
smooth surface.

This work has been described in a series of key stages, rather than pages of equations. Important
factors are the geometric details of stretching a string, and a correct yet useful treatment of
elasticity. It is worth giving an example of a relation having both these theoretical aspects, and
also some practical interest. This concerns just the stretching from pressing down between two
frets of equal thickness. The effect can be appreciated on a modern lute by carefully stopping a
fret, without pressing behind it, plucking the string, then pressing down and hearing the pitch
rise. The equation is

\[ L \frac{t}{(L - l)L} = \frac{m^2(E_s/E_0)}{(l - l_n)s_0} \]

L is the length of a bass string with no stretching effect; t the extra length needed for a treble
string;
l is the position of a fret from the nut; l_n the position of the fret nearer to the nut;
m is the height or diameter of a fret; s_0 the initial stretch of a slack treble string up to its nominal
pitch;
E_s is the average tensile modulus in reaching this pitch; and E_0 the modulus near s_0.
This relation can be used to find m, t, (E_s/E_0s_0), or compatible values. Similar relations for other
portions of a string need to be added for the total stretch effect. In addition to the previous main
conclusions, some other details can be seen from the equation.
(i) Double frets, as in the Holbein painting, effectively increase l_n, but the result is negligible
unless the two strands are opened widely.
(ii) Tuning temperaments set for a meantone might be pushed towards equal temperament,
because the naturals have greater sharpening. This results from a smaller (l - l_n), because l_n is
greater for the flat enharmonics.
(iii) The detailed measurements of stress functions, referred to in section 3, showed that the
different sizes of monofilament and metal-wound strings have similar initial moduli. Also, the
stretch sharpening of a wound bass is similar, about twice, that of a treble string. Although the
treble has a much higher initial stretch s_0, which would give much lower sharpening, the
modulus E_s has also increased at a higher position up the stress curve. An example of this
important general effect is seen in the equation.
(iv) This local behaviour of a treble suggests a useful possibility for bass strings. Only a limited
region of low modulus around the operating stress might be needed. Some controlled stick-slip
may be possible in a construction of several strands in contact. An abrupt effect was noticed in
the tests on a metal-wound bass, presumably between the winding and core.

(v) Practical tests could be carried out on this scheme that appears necessary to explain shorter
bass strings. If it is difficult to arrange angled bridges, then small spacers could shorten lower
strings. If highly elastic bass strings cannot be obtained, then much stiffer trebles may be
sufficient at first. A thin, twisted Dacron has enormous strength and very small stretch. This was
originally used in textiles, and has replaced nylon for very long fishing lines. Testing the reverse
fretting would need carefully chosen fret diameters and heights of bridge and nut.

(vi) Not all early lutes had bridge angles resulting in shorter lower strings. A smaller number in
Ref 3 had no angle or an opposite angle.

N.B. This implies a need for an even greater variety of strings and frets with extreme properties,
and may be seen later as part of a ‘reductio’.

(vii) Modern lutes have unangled bridges, but might benefit from slightly longer bass strings, as
used on some guitars. In addition to all the detrimental effects of a low elasticity, the end
corrections for a sounding length, including the loop at the bridge, are larger for the bass strings.
About 2mm might be useful, which makes the effective difference from old longer trebles even
greater, and perhaps as much as 8mm.

Finally, after suggesting how lutes with shorter basses could have been tuned, an obvious
question remains. Why was this system used, and also why bother to reinvent it, if good tuning
can be obtained on modern lutes with present synthetic or gut strings. One reason, together with
the advantages of high elasticity given in section 2, may be that the old system appears to aim
for stability of tuning in the bass, and flexibility or expression for the treble. In contrast, the
properties of a modern lute have the opposite tendency. This would indicate an elaborate use of
natural properties and geometry by the original makers, for a musical purpose. Modern
mentality tends to aim at reducing defects, often with a gain in power and loudness but a loss of
colour in several orchestral instruments.

Addendum 1. Metal-wound strings

This section contains several points specific to strings with a winding of metal wire. These
strings with an increased effective density have a core of gut or synthetic material, and have
been the main type of construction for modern bass strings.

The tests in section 3 showed that the low F Pyramid had a relatively high effective tensile
modulus, comparable with the middle f monofilaments, and the initial value for the treble f’.
Since the winding can contribute little, this indicates that the thin nylon core surprisingly has up
to four times the tensile modulus of solid nylon. This might indicate little effort to make bass
strings with equal or greater initial stretch than the higher strings. However, there would be a
practical limitation on the initial stretch, in order to avoid irreversible disruption of the closely
wound wire. A more open winding or another way of adding mass might reduce this problem.
At the same time, all metal-wound basses with synthetic cores have a low flexural stiffness or
rigidity, resulting from the narrow core made of fine separate strands. This produces a well-
known unhistorical, and perhaps undesirable, low decay or twang. Both these problems, of high
tensile stiffness and low flexural stiffness, might be improved for this type of string designed
initially for modern guitars. Cores could have a larger diameter, and use a lower modulus
material in a single filament, similar to an a or d’ string; or far fewer strands, perhaps twisted
together.

Modern basses using a gut core, even if well made and highly twisted, can have relatively low
initial stretch or high tensile modulus. Also, their thicker core can produce stronger damping,
which may be more historical and attractive. This suggests that the above ideas for synthetic
strings are practically useful, and may have the further benefit of higher initial stretch.
The effect of a winding on elasticity will vary with each design and maker. A thick, stiff wire that grips the core, perhaps made by winding onto a stretched core, will have more effect than a fine, soft, looser wire. During the test on the F string, a larger stretch up to an A, above the normal range, gave an alarming slip. This was probably a release of grip, and suggested the general idea of a local low modulus in section 6 (iv). All the relative effects of mass, density, diameter, tensile and flexural rigidity can be expressed and discussed more easily in equations. Here, one practical result is that the flexural stiffness can be expected to decrease inversely with the number of strands in a core. This would not greatly affect tensile stiffness, but it is well known that further twisting together of strands is a major way of reducing the tensile modulus. However, this twisting could then impair the flexibility. Any locking or adhesion between strands could increase both types of stiffness. Also, for any type of F string, typically having a diameter of 1.0mm and a length of 600mm, at a tension of 3kg, a required effective density of about 3gm/cc can be calculated from part 3 below. This shows that the volume required for a typical metal is only a quarter of the total. This gives plenty of scope for designing useful cores, windings, etc.

In this digression to modern strings, the recent development of Nylgut, and any similar synthetic polymers, can be viewed in terms of the elastic modulus criterion. This single material has a smaller modulus than nylon, and a greater density close to natural gut. It supplies a rather thin treble string, or a thicker one at an increased tension, but good 2nd and 3rd courses, and a much-improved 4th course, where nylon is unusable. It can even provide a reasonable 5th course, but for larger diameters of lower strings, problems are produced by the lower stress, low initial stretch and larger bending force. From the modulus criterion, and the example of this material, it can be seen how materials and constructions with a graded decrease of modulus, and increase of density, could be more useful than a single material, or the abrupt changes in a mixed set of nylon, Nylgut, PVF, and metal-wound strings. It might even be possible to avoid metal-wound bass strings. It may be undesirable to remove interesting tonal differences between strings.

N.B. Part 3 extends this discussion, and part 4 contains a later practical suggestion for a more elastic single material on some lower courses.

Addendum 2. String diameter and density
The familiar practical string parameters of tension, diameter, nominal length, and density were scarcely mentioned in the main paper. They could be regarded there mainly as dimensional factors, and the modulus criterion was usefully independent of tension and diameter. However, the conditions of elasticity and fretting, which were found to be necessary for an explanation of bridge angles, began to appear far too intricate and variable. After analyzing the details of fretting, and seeing their importance, it occurred to me that another variation along a string might be able to explain angled bridges. From the theory of vibrating strings, variations in the diameter and density appeared to be the only possibilities. This is a considerable wrench for our modern mental conditioning by precision engineering.

Starting with the previous reference of a bass string with no stretch sharpening, ideal frets, and now a constant diameter and density, I realized that a treble string that decreased in diameter and/or density, from the nut towards the bridge, could explain a bridge angled to give longer treble strings. An initial calculation showed that a taper of only 2% could explain the 3° angle, or a 3mm longer treble string. A general relation for the extra length of the treble is

\[ t = \frac{1}{4} L \left(1 - d_b/d_n\right) \]

Here \(d_b\) and \(d_n\) are effective diameters at the bridge and nut. The basic reason for the effect is that for higher frets the tapering leads to a sounding length of a smaller average diameter, and hence a higher pitch, since frequency varies as 1/d. There is also a simple geometric grading of the effect with sounding length, or fret position, unlike the previous complicated stretching.
A varying diameter can also account for differences in density since the fundamental parameter is mass per unit length. For example, if there is a constant diameter \( d_c \) with a varying density \( \rho \), then an effective varying diameter \( d \) can be defined by \( d^2 \rho_c = d_c^2 \rho \), where \( \rho_c \) is an average constant density. The effective diameter is

\[
d = d_c \sqrt{\frac{\rho}{\rho_c}}.
\]

A tapered string could be a statistically likely result for an old manufacturing process that attempted to make a uniform string, and guts are also naturally tapered. A neat way to grade the effect across six courses might start with a long tapered string, then take out appropriate sections; and perhaps even reverse the taper for lower strings, to enhance the relative effect. Bass strings might be produced from several thinner basic tapered guts, and hence could be made less variable. The amount of taper could be controlled, by reversing some component guts. The characteristic amount of taper, about 2% for a 3° bridge angle, might imply some average condition above which strings were discarded as too difficult to fret, or false for a number of reasons. In addition to the large-scale taper over about 10cm, natural strings would also have non-uniformities on a smaller scale of 1cm affecting fretting, and below 1mm affecting damping and tone. All scales could give a rather random inharmonicity of overtones, in contrast with the progressive sharpening from elastic bending.

At this point I recalled historical reference to tapered strings (Ref 4), where Capirola, or his student Vitale, had written:

'Strings are made from the guts of castroni. They taper, so you must tie the fifth and sixth courses with the thick end at the bridge and the second, third and fourth courses with the thin end at the bridge. Strings from Munich are more even in thickness. Strings must be true, but if slightly false the two strings of the course must be matched - a true string and a false string just make a false course. Sometimes you can solve this problem by turning one of the strings round.'

This has a clear but incomplete correspondence with the analysis, similar to the fretting in section 6, and raises further questions. The predicted tapers were very small, perhaps discernable mainly from the geometry of a gut with its end pieces. The advice to turn round a gut suggests some effects were slight, and may also have involved density variations. The mention of only taper, and no other property, for fifth and sixth courses may imply that all his strings were made by the same process of simple twisting, rather than a different complicated roping or braiding for the basses. The reverse taper for courses 5 and 6 indicates a distinction from courses 4 to 2, rather than between 4 and 3. The Holbein painting also shows much thicker, and smooth, fifth and sixth course basses, as distinct from their thin octave strings and the higher strings. The important first treble course is not mentioned, which may indicate a special reference elsewhere, such as tuning close to breaking point, or a Munich string. Neither is there explicit mention of octave strings. The reference to Munich strings seems to indicate a preference for more uniform strings, while making best use of available tapered strings.

Another way to understand the use of tapered strings is to take the analyses of tapering, fretting and elasticity; a bridge without an angle; and the basic tapered gut strings implied by Capirola. The fifth and sixth course basses with their thicker ends near the bridge can be a clever way of flattening considerably sharpened fretted notes caused by insufficient elasticity, or excessively high modulus. The overall purpose here is similar, but with important differences, to the longer bass strings of some modern guitars.

The uses of string taper considered here can work in the same direction as the main analysis of elasticity and frets, towards an explanation of how shorter bass strings were tuned. However, it is not possible to distinguish whether the two mechanisms were alternatives or used together. Many possible combinations of elasticity, taper, bridge angle and fretting are conceivable. It seems wiser to bear in mind the various factors until further evidence and analysis. The variety of combinations may be reflected in the different bridge angles noted in Ref 3, perhaps even at the same locality and period. The modern approach to authentic stringing has produced some
complicated designs for which there is slight evidence, but the survival of angled bridges and Capirola's instructions on taper deserve some practical attention. Finally, in Ref 3 some old lutes had systematically angled bars, in addition to an angled bridge. This may modify vibrations across the soundboard, which will have a low frequency as a result of the low stiffness across the grain. It may also be possible to say something useful about soundboard thicknesses, and driving of the bridge by the strings.

N.B. About two years later, after most of the work on soundboards, this nagging problem of shorter bass strings was revisited and is resolved, hopefully, in the next communication. Four key factors were the implications of the elastic scheme and a loosened attachment to it; coming across angled necks as well as bridges; a close reading of Capirola; and an estimate of natural tapers. This addendum shows the earlier close of play.

References

3. THEORY FOR ELASTIC STRINGS

In the previous main paper, part 2, it was hoped initially that detailed theory would not need presenting in order to reach a practical relation between string elasticity and bridge angles. The later realization of the possibilities for tapered strings greatly complicated the issue. This suggested that the other useful aspects of elasticity, such as the modulus criterion, merited some definite explanation.

For a string of length $l$ and a uniform mass per unit length $\mu$ at a tension $T$, the velocity of transverse waves is $c = \sqrt{T/\mu}$, and the wavelength of the fundamental standing wave is $\lambda = 2l$. The frequency of this first harmonic is

$$f = \frac{c}{\lambda} = \frac{\sqrt{T/\mu}}{2l}$$

This is often treated as a constant in much applied maths, where the tension is assumed constant within certain small limits, and the transverse amplitude is small.

Since $\mu = Ap$, where $\rho$ is the density, and $A$ is the cross section area $\frac{1}{4}\pi d^2$ for a string of diameter $d$, the frequency can be written as

$$f = \frac{\sqrt{T/\rho}}{d/l}$$

This practical relation, with some testing and experience, is used for selecting musical strings from a maker's range. For a given material, $\rho$ is known, $T$ and $l$ are chosen, and hence the required pitch $f$ determines the necessary gauge $d$. However, with four variables there is great scope for a more general choice, and yet this relation gives no clear way of deciding desirable properties for a string material. A more basic relation is

$$f = \frac{\sqrt{\sigma/\rho}}{2l}$$

where $\sigma$ is the stress $T/A$ in the string. Stress is a physical state in the material, and also the particular dependence on $d$ and $T$ has been eliminated. The diameter $d$ in the relations refers to the stretched string, so that measurements of frequency can give a true stress. However, the initial choice of string diameter should allow for the small decrease while stretching up to pitch.
The basic relation for \( f \) also shows that strings of the same material and length but different diameters may break at the same frequency, since failure occurs at a critical stress \( \sigma_b \).

The initial stress \( \sigma_o \) in a tuned string is reached on the stress-strain relation \( \sigma(\varepsilon) \) by stretching an amount \( x_0 = \varepsilon_0 l \). The strain \( \varepsilon_0 = x_0 / l \) is also a basic state in the material. The stress at \( \varepsilon_0 \) is found by summation or integration along the stress-strain curve over the range \( \varepsilon = 0 \) to \( \varepsilon_0 \):

\[
\sigma_o = \int_0^{\varepsilon_0} \sigma(\varepsilon) \mathrm{d}\varepsilon
\]

Here \( E(\varepsilon) \) is the elastic modulus or slope \( \mathrm{d}\sigma/\mathrm{d}\varepsilon \) along the curve. For the simple case of a linear relation, \( E \) is a constant so that \( T/A = \sigma_o = E\varepsilon_0 = Ex_0 / l \), which shows the connection between modulus, tension and stretch.

For many reasons, outlined in part 2 and the letters, it is desirable to have a large initial strain or stretch. In this way any further increases in strain \( \sigma_1 \), that occur on plucking or pressing a string down to a fret, do not increase too perceptibly the stress, and hence the required pitch. For example, if a pitch increase as great as a fifth of a semitone were just acceptable, \( \sigma_1 \) would need to be less than 2% of \( \sigma_o \). A related effect is a lower sensitivity of the initial tuning to winding a peg. The new stress \( \sigma_n \) at the increased strain, \( \varepsilon_n = \varepsilon_0 + \varepsilon_1 \), can be found by a new integral, or by an approximation. The simplest form would be

\[
\sigma_n = \sigma_o + \varepsilon_1 E_0 \varepsilon_1
\]

where \( E_o \) is the modulus or slope near \( \varepsilon_0 \).

The increase in strain \( \varepsilon_1 \) during playing will be similar for all the strings on a lute, but slightly more for lower strings with a higher action, and varying with frets as in part 2. Therefore, a useful goal for a good set of strings is that the initial strain \( \varepsilon_0 \) should be large and roughly equal for all the strings. The frequency relation can be written as

\[
f^2 = \int_0^{\varepsilon_0} E(\varepsilon) \mathrm{d}\varepsilon / (4\pi l^2)
\]

over the range \( \varepsilon = 0 \) to \( \varepsilon_0 \). This gives a condition for the elastic modulus of each string with a desired pitch \( f \). For a string of length \( l \), some measure of the modulus divided by density should be proportional to the square of the frequency. This is the modulus criterion suggested and discussed in the other parts of this communication. Some special cases can illustrate this idea.

If \( E \) is a constant, so that the stress curve is a straight line, the various moduli divided by density, simply need to be proportional to \( f^2 \) for each string.

Next, if \( E \) is an arbitrary function or relation of the form \( a y(\varepsilon) \), then since \( y(\varepsilon) \) is integrated over the same range of \( 0 \) to \( \varepsilon_0 \) for each string, the different constants \( a \) for the various strings just need to be proportional to \( pf^2 \). This scaling is remarkably simple and could be useful.

The modulus criterion can also examine what happens when the same material is used for different strings. In practice, two or even four courses commonly use the same material. Perhaps surprisingly, this analysis is more involved, even for a constant modulus \( E \) where

\[
f^2 = \varepsilon_0 E / (4\rho l^2)
\]

For a more extreme case of a two octave interval between sixth and first courses, \( f \) increases by a factor of 4 so that the strain \( \varepsilon_6 \) of the bass string, or its stretch, is only \( 1/16 \) of the strain \( \varepsilon_1 \) for the treble. This extreme breaking of the criterion can cause problems for all single materials. The stresses are in the same ratio as the strains, and if the tensions are about equal, then the diameter of the bass needs to be 4 times the treble diameter. This must also hold independently of the stress-strain relation, from the second practical equation above.

A simple linear relation is not very realistic, and a more practical stress curve may have the form \( \sigma = \alpha \varepsilon^n \). For a single common material this leads to \( \varepsilon_1 / \varepsilon_6 = 16^{1/n} \), and the modulus also increases up the curve as \( E_1 / E_6 = 16^{(1-1/n)} \). For example, a quadratic approximation with \( n = 2 \), where the
curve becomes steeper as in a real material, gives $\varepsilon_t/\varepsilon_b = 4$, and $E_t/E_b = 4$. This is exactly midway between the linear relation, and the ideal modulus criterion for different materials with $\varepsilon_t/\varepsilon_b = 1$, and $E_t/E_b = 16$. It also follows that an extra strain $\varepsilon_j$, from plucking and fretting, has more relative effect in the bass than in the treble, but not so severe as the simple linear case with a constant modulus. For larger values of $n$, and stronger increases of $\sigma$ and $E$ with strain, there will be more sensitivity to increases of strain $\varepsilon_j$ for the treble, and a decreased effect for the bass. In theory, a very high value for $n$ approaches the ideal criterion with equal effects. However, as seen below, such values of $n$ do not occur in practice.

More complicated forms of $\sigma(\varepsilon)$ would require careful treatment, but the idea of an effective modulus may still be useful. A difference of density between strings is straightforward, and in practice an increase of density for bass strings is the familiar alternative to large reductions in the modulus, increases of diameter, inflexibility, etc. The above relations are not found in applied maths work on strings, which uses the assumption of a constant tension. In a sense they are a systematic inversion of the usual frequency relations, to focus attention on the string properties rather than the geometry and applied tension.

These formulations can be used for detailed analysis and description of any type of string. As a short practical illustration, the tests in part 2 on the nylon treble f' string gave an initial stretch of about 35mm. The effective value of $n$ was about $1\frac{1}{2}$, midway between the constant modulus and a quadratic stress-strain curve discussed above. The middle f had a low initial stretch of about 8mm, and the stress curve was linear over this small range, but a little higher than for the actual material used for the treble. The more useful, thinner PVF string, and also the metal-wound F string, needed a larger 11mm initial stretch.

There are some further important points about tests on stretching, pitch and stress. Degrees of stretch sharpening can be expressed simply as a stretch, measured in mm from a marker on a string, needed to produce say a semitone rise. This can be used to make simple comparisons, as in paper 1. However, if values of stress are required, then musical intervals must be converted to frequency ratios, or frequencies in Hz, and then squared, as in the third equation above. For a stress-strain curve, these values can then be plotted against stretch $x$ divided by $l$. With larger values of $x$, the strain is given more accurately by $x/l(l-x)$. Absolute values of stress can be found from the second equation, manufacturers' tables, or direct measurement with weights. Intervals are sometimes simply plotted against stretch. These curves will have a log scale with a decreasing slope, as if $n$ were less than 1, and look misleadingly like yield curves for a metal.

In part 2 many calculations were made of the extra stretch for fretted strings. For a string of length $l$ being plucked at a distance $q$ from the bridge by an amount $p$, the extra stretch of the portion next to the bridge is just

\[(\sqrt{(q^2 + p^2)}) - q \equiv p^2/2q\]

The portion next to the nut has an extra stretch of $p^2/2(l-q)$. These relations are well-known for harpsichords and can also be used for a lute. With typical values the extra stretch is about 0.1mm. For a fretted string, similar relations were used in part 2 and paper 1 for the two portions of string between the fret and bridge, and between the lower fret and nut. In addition, the short length of string, under the finger between these two frets, was treated by two similar relations, and an example given in part 2. All four parts were added to give the total extra stretch, which was needed to calculate the increase of stress and pitch. Plucking could also be included for a fretted string.

This theory may form useful background for the other work on strings, but treatments of all the other topics would be lengthy. Conveniently, inharmonicity caused by elastic bending can be outlined later in paper 5 on the physics of vibrating soundboards. Some aspects of non-linearity are treated in relation to tone quality of upper strings in paper 7.
4. A MODIFIED STRING MATERIAL

The analyses in the previous sections can be applied to suggest a modified synthetic material. This would be like Nylgut, or any equivalents, but with about half its elastic modulus, or twice its stretch. Nylgut is a polymer with a density close to natural gut. The new material would be expected to be useful for a low sixth course G string. Present Nylgut is probably good down to only a fifth course C, although metal-wound strings are normally used here. Analysis indicates that for a material to work an interval of a fourth lower, its modulus needs reducing by a factor of 0.56, or (3/4)².

This suggestion should enable stringing a basic six course renaissance lute in two simple materials. The new material would also improve the fifth and fourth courses. It could serve for a third course, and also a second. With brighter nylon on the top course(s) there might even be no need for present Nylgut. A single material for the whole two-octave range of six courses would require a fourfold change of string diameter, giving an inverse sixteen-fold variation of stress, with a roughly similar change of initial stretch, which may be too demanding. At the other extreme, a different modulus would be ideal for each string, but this might be difficult for a product range.

The new material might supply slightly lower bass strings, but their increased diameter may indicate a denser polymer, or a little metal winding or loading on a thick core. For example, a bass D string with the diameter of a G string would need an average density 1.8 (or 1/0.56) times that of Nylgut. This is the inverse of an alternative further decrease in the modulus of Nylgut down to a factor of 0.32, or (0.56)².

The sound and feel of different strings is a matter of taste and instrument. For example, in my limited experience, one lute is nice with nylon, PVF and Pyramid metal-wound strings; and also good with Nylgut and wound Kurschners. Another lute is poor with the former scheme, but sounds nice with the second. This increased my practical interest in Nylgut, and in removing or delaying any transition to metal-wound strings, whose sound and feel may not be entirely attractive or historical. The only present alternative seems to be one or two disproportionately expensive natural gut constructions for the lowest strings, costing more than the other dozen together.

It is well known that nylon is useful down to only the third course, but it could still be used for the first course, and even the second, in preference to the large initial stretch of Nylgut. The real choice would be between Nylgut or a modified type, and the metal-wound strings with PVF that are normally used below nylon.

The new strings would look similar to renaissance paintings with pale, smooth strings, increasing in diameter across the six courses, and octave strings beginning at the fourth or a lower course. One reply to this suggestion might be gut strings throughout, with fine sound and more authenticity. The present aim is a material close to gut, but more consistent, durable, stable in tuning, available and affordable. If this development of new materials is difficult, perhaps present Nylgut with some clever twist could reduce modulus; or an open winding, loading or PVF could increase the density. The sound might be further enriched, by introducing slight irregularities along a string.

N.B. A possible problem with this suggestion may be inharmonicity for a thicker monofilament. This can be estimated from the theory in paper 5. The small term describing this effect for a present fifth course would be enhanced by a factor of (4/3)⁴, due to the larger diameter, but reduced by (3/4)² from a new lower modulus. It is unlikely that the overall increase of 1.8 would be crucial but these details need experiment, as explained in paper 5. Separate twisted strands or added mass might reduce any effect.

Finally, it should be stressed that this scientific interest in strings does not disregard the skill and effort required for practical developments, nor the detailed recommendations for currently available strings from experienced suppliers and lute makers.
TAPERED LUTE STRINGS, ANGLED NECKS AND BRIDGES (3)

Modern lute players, whether using gut or synthetic strings, may have wondered what it was like for Capirola, da Milano, Dowland, or the Gautiers. As described in the previous communication, I had been attempting to understand ancient lute strings, in addition to details of modern strings. This involved physics, a little biology, details about old lutes, written sources and practical experience. The major, and hardest, part has been working out physics that is compatible with the old lutes. As this came into place, the other evidence became meaningful, leading to implications for the earlier development of strings and lutes. This account includes all these aspects, and begins with the physics because this deduces some important properties of the old strings, of which there may be no surviving examples. The earlier research was described in paper 2, and this much later work may form a resolution of the roles of elasticity and taper.

1. Elasticity
Slightly later than the composer-players another genius, Robert Hooke, formulated elasticity. It is suggested below that the original inventor-makers of strings and lutes had already mastered this material property, in ingenious ways that may only now have come to light. Elasticity allows us to wind strings up to pitch, then pluck, produce a good tone, and have reasonable fretting. This can be difficult on a lute, with its wide range of notes, but only a single compromise nominal string length, and simple frets. Problems are caused by the low elasticity of the bass strings, so it is increased.

Nowadays, all of us are unconsciously conditioned by engineering, precision, and the removal of defects. The necessity of using natural materials to advantage has declined. This can become a problem when trying to understand old artefacts, as seen later, but I shall avoid 'old good, new bad'. From the physics I have concluded that modifying elasticity does have very important uses, but for the more puzzling aspects of old lutes it is not a possible mechanism.

2. Tapered strings
An alternative property was needed, and I could see that the only possibility in the theory of vibrating strings is the distribution of mass. The simplest departure from perfect uniformity is a string with a tapering diameter, which is not expected by our modern conditioned mentality. I have analysed this novelty mathematically, and found that barely detectable tapers in the region of 2%, as distinct from say 0.2 or 20%, can give consistent, widely applicable solutions to all the difficult puzzles. A 2% taper is equivalent to a 1.00mm diameter increasing to just 1.02mm over an entire string. To be clear, in practice this is a long-range trend, like a rocky slope rather than a perfect ramp. A modern uniform string might be a still pond. Also, cross sections may not be exactly circular, and stretching could change these irregularities in natural materials. A variation in effective diameter can also account for a variation in density. The initial analysis formed an addendum to the earlier paper 2 on angled bridges.

Several effects will now be examined, beginning with the areas where good elasticity is needed, and then
(i) a basic effect of taper on stretch sharpening,
(ii) a remedy for guitars,
(iii) a complete explanation of the taper mechanism,
(iv) a reversal of taper,
(v) the elasticity of modern strings, and
(vi) a final amazing use of taper.
These topics represent practical situations, in order to reduce excessive abstraction. An important aspect is comparing very different strings, such as basses with trebles, as seen in paper 2. The lack of modern tapered strings is the greatest problem for practical appreciation, and the reason why theoretical science has been necessary. Making sketches may be helpful, but if taper initially seems too obscure or unnecessary, then sections 4 and 7 could be a helpful start.

3. Combined effects of taper, elasticity and string length
Adequate elasticity is needed for initial stretching, and the tuning of open strings. This also helps a good tone rather than a short dull sound. It can also reduce the sharpening effect from stretching a string when plucking, and also when fretting a string. It can be difficult to obtain sufficient elasticity in the lower strings. The stresses are much lower than in the trebles, about $1/16^{th}$. This is because the tensions are about equal but the cross section area of bass strings with the same density is much larger, by about 16 times for a sixth course. The diameters are about four times greater because the sixth course is two octaves lower than a treble string. Reasonable degrees of stretch, to achieve the four conditions above, therefore require very high elasticity, or a low modulus. All these matters were covered in paper 2. Taper is now explained, under the six topics listed above.

(i) Even if the first three requirements of elasticity are adequate, there may still be significant stretch sharpening by the frets. I have analysed how string taper can completely eliminate this problem. Consider a uniform diameter bass string which stretch sharpens. If the length of string between fret and bridge is made thicker, then the note can be flattened and corrected. A gradual taper just corrects more accurately on all the frets. The following points will provide further quantitative explanations.

(ii) For uniform bass strings without sufficient elasticity, as on many modern guitars, the stretch sharpening can be reduced by making the basses slightly longer, by typically about 3mm, using an angled bridge or individual saddles. This works by causing extra flattening on higher frets. Modern lute basses might also benefit, but see later. It is worth noting that bandoras are not relevant here.

(iii) The basic geometric effect of taper can be seen from a string that is ideal, or without any stretch sharpening. Consider a uniform ideal string with correct frets, and also a tapered ideal string of equal length using the same frets. If the thicker end is at the bridge, the notes on higher frets become flatter, relative to the uniform string, because the sounding lengths become thicker. This can be corrected by slightly shortening the thick end, with for example an angled bridge. The analysis given in paper 2 showed that the percentage shortening needed is about one quarter of the percentage taper.

A detailed derivation is appropriate for this key effect. It can be explained with a string of diameter $d_b$ at the bridge, decreasing to $d_n$ at the nut end. The average diameter of the open string $d_o$ is therefore $1/2(d_b + d_n)$. Now take the twelfth fret, for the octave of the uniform string. The diameter here is simply $d_b$ and hence the average diameter $d_s$ of this sounding length is $1/2(d_b + d_s)$ or $(1/4 d_b + 1/4 d_s)$. The basic frequency for a string of length $l$ is proportional to $1/d_l$, where $d$ is now its average diameter, which is valid for slow variations over a large length. From this it can be seen that a small decrease of length, $t$, at the bridge end of the tapered string will increase the pitch at this twelfth fret more than for the open string. The required condition is

$$\frac{(l-t) d_s}{2} = \left(\frac{l}{2} - t\right) d_s$$

The three relations can be solved to give the value of $t$ as

$$t/l = \frac{1}{4} \left(\frac{d_b - d_n}{d_b}\right)$$
This is the result described above and in paper 2. From this analysis, a 2% taper requires a shortening of about ½ %, which for a 60cm long string is 3mm. In this example with simplified numbers, the correcting effect in (i) can be seen as just the cancellation of the low effective elasticity in (ii) by the taper in (iii). It is convenient here to use the extra bass length in (ii) as a measure of stiffness.

In greater detail, for lower frets the initial factor increases from ¼ to ½. The general result for a fret with a frequency ratio \( r \) is

\[
t / l = \frac{1}{2} \left( d_b - d_a \right) / (r d_b)
\]

The correction is not an absolute constant, but the variation about an average factor, of 3/8 for a fourth fret, is small. There are also some far smaller terms of order \( td \) that need consideration.

(iv) Reversing the previous taper, so that the thinner end is at the bridge, requires a lengthening at this end in order to fret as a uniform string. Another use below is making treble strings fret sharper, to match lower strings which have more stretch sharpening. This is the reverse of (i).

(v) As a reference, we also need the conceptually simpler case of trying to make basses very elastic, so there could be no need for lengthening or taper. This is the modern approach, which has inherited two centuries of a different string technology, between the decline of the lute and its recent revival. Basses with metal wire wound onto a thin core may not be made highly elastic without disturbing the winding. These basses, and also solid gut basses with elaborate twisting, usually have more stretch sharpening than upper strings, partly also caused by a higher action. The next points are more difficult to grasp. The analyses in paper 2 showed that large changes of elasticity would be needed just to equalize this effect of stretch sharpening for bass and treble strings, as might be desirable for equal length strings. A 16-fold increase of elasticity from a first to a sixth course could be necessary. Further increases would be needed to reverse even a slight effect, so that trebles had slightly more sharpening than basses. A complete reversal might require a 256-fold increase of elasticity in bass strings. The strangeness of this fanciful situation can be visualized on one's own lute.

(vi) Finally we come to the most curious and important case. Here, many old lutes have fretted lower strings that are considerably shorter than the trebles, as a result of angled bridges, and also angled necks. The analyses in paper 2 can be seen as a demonstration that this cannot be achieved practically for uniform strings by increasing the elasticity of basses, or decreasing it for trebles. Even if a special case could be devised with a "high strength dense rubber", to provide the '256-fold reversal' in (v), all the intermediate courses of a lute would need grading. In addition, different angles or length differences would require completely new properties, and also for a range of different lute sizes and types. This case can be seen as an extension of (i) or (v) with a greater degree of taper, and the resulting use of the frets is like (iii) and (iv). This was a hard won and very useful result.

A potential complication of taper might be inharmonicity of overtones. Ancient tests for falseness of strings, referred to later, can be taken by us as direct experiments showing that inharmonicity and defects were a great concern but did not prevent musical use of many strings. Also, I have calculated that the nodes shift slightly compared with a uniform string, and the overtones appear inharmonic even more slightly. There will also be larger local irregularities, which may add colour, and moderate scale effects on some random groups of fretted notes, which could be good or bad for detailed tuning. The basic effect of an irregularity on inharmonicity can be seen from imagining a small mass added to the mid point of a uniform string. The frequency of the fundamental is lowered, but the first overtone is unaffected since the mass is now at a node. The effect on fretting can be understood from a small added mass just behind say the fifth fret. The effect of lowering a fundamental would be greatest for the open
string, and decrease at each fret until the fifth fret excludes the mass. As more irregularities are added along the string, the fretting could become more uniform.

An interesting possibility is that some effects can cause a flattening of overtones. This might be useful in tending to cancel the progressive sharpening caused by the more familiar elastic bending. The final test is the instrument and ear, especially since theory will become very sensitive just where the effects become important, as seen in paper 5. Many hypothetical combinations of irregularity with taper can be analysed, but there is no present definite purpose. It has sometimes been noted generally, and in old writings, that the variability of old strings could require frequent tuning adjustments by the movable gut frets. However, there does not appear to have been any definite model or use of non-uniformity until this analysis of taper.

If this unfamiliar theory has seemed abstract, it is still useful to see that the above percentage changes of length and diameter translate simply into tuning effects of order a tenth to half a semitone. Suitably used, these can compensate each other and poor elasticity, but bad reinforcements would make tuning impossible.

The following examples and analysis may show how the old makers sidestepped elasticity. For us it is like trying to push a large stone up a hill - the problem of elasticity; and then finding a two-way escalator - the taper. In the historical examples, some descriptions will seem familiar, but now causes can be seen. These are surprising and may take time to digest. It would be unwise, however, to be too certain since there will always be new evidence.

4. Capirola’s six course lute

Lute historians know that Capirola’s Lute Book of 1517 (Ref 1) discusses the use of tapered gut strings. In my case this was recalled and its significance realized, only after seeing the physics. Using the analysis in cases (i) and (iv) above, I have worked out a definite scheme from his ‘secret of stringing the lute’, most of which has become understandable and useful. The result is an elegantly clever way of compensating for stretch sharpening with the available tapered strings. This stretch sharpening seems significant even for his fourth course bass, which may indicate a rather simple inelastic gut, in modern terms. We start with a sixth course bass, which is strongly tapered, maybe more than 2%, with its thicker end at the bridge. Then I deduce a decreased degree of taper on the fifth course bass, where there would be less stretch sharpening for a thinner string. The fourth course bass has a small reversed taper, and the upper three courses have slightly increasing reversed tapers. The octave strings, which are implied on the lower three courses, to help a dull tone, would have appropriate reversed tapers. The frets may be viewed as set for an imaginary uniform string with stretch sharpening, between the fourth and fifth courses. Therefore, the lowest two basses use case (i), and the other strings use case (iv) in section 3. The strings were most likely of equal length, but taper is so useful that variations with angled bridges could be accommodated easily, unlike uniform strings.

Capirola used ordinary tapered strings and a ‘da ganzer’ variety, related to dressmaking. He also referred to strings from ‘Monaco’, probably meaning Munich, which only sharpen whichever end is at the bridge. At first sight this may indicate a uniform string, but the strict meaning seems to be a taper no more than is needed to cancel its own stretch sharpening. A sales slogan could have been ‘the strings that never flatten’. Their maximum sharpening, when the thin end is at the bridge, would be limited to twice their stretch sharpening. Strings with this property could be used for the fourth course bass; surprisingly the fifth course bass, needing only slight relative flattening; and perhaps even the sixth course bass. (This was not my initial thought, and maybe not a final one)

The ‘M’ strings could also provide small tapers for the third course, where Capirola warned against having a thick end at the bridge. This advice would be useful because the third course is probably a tempered third above the almost uniform fourth course bass. The third course has
‘sharp’ types of enharmonics, such as a Cº on the fourth fret. These are already flatter than ‘flats’ and must not be flattened further by neglecting to reverse the taper after the fourth course. At the second course more reverse taper is needed, again for a thinner string with less stretch sharpening. If the stretch sharpening were less than about half the amount for the fourth course bass, M strings would not have sufficient taper, from the properties deduced above. This would be even worse for the first course, and strikingly, Capirola did not mention the important first course. Perhaps he recognized a superior general quality, consistency and strength of M strings, as referred to later, but found their taper too limited for his scheme. This discussion of M strings is also needed for treating later lutes.

Sometimes Capirola appears literally contradictory in the translation, but one can see this is a result of trying to combine understanding and trends with practical instructions, and the physics helps to unravel this. Much more can be deduced from this hitherto mysterious source. Historians of science might be interested in this early acoustics.

5. The seven course lute
An increase in elasticity for lower strings was likely during the sixteenth century, and an exciting support to composers of complex music, including da Milano. By 1610, John Dowland (in Ref 2) suggested unison sixth course basses, without an octave, so tone as well as fretting must have been good. Such improvements would have led to the seventh course, at first a tone and then a fourth lower, with an octave string, and also fretted. Taper could provide good tuning, even if tone is not perfect, with a lower elasticity than needed for uniform basses. In the various English sources acoustic evidence is thin, particularly for bass strings. These could be very strong, needing replacement far less often than fragile upper strings.

The critical and expensive trebles accounted for half of Dowland’s total advice. These music books mainly provide consumer advice; names and colour codes for string types and their origins; and warnings of low quality rotting strings. The content and purpose is different from Capirola’s tuning instructions. There is no explicit mention of ‘taper’, but Dowland suggested thicker ‘gansars’ on second and third courses, as previously noted by Capirola, and also strings from ‘Monnekin’. Le Roy had preferred strings from Munich in 1574. Much later in 1676 Mace advised ‘Minikins’ for the first three courses, and octaves down to the sixth course (Ref 3). Most of these M strings look like Capirola’s with a limited degree of taper. This is an important distinction from a uniform string, which may have been assumed previously. Now the reference point could be between the third and fourth courses, and able to provide sufficient taper for the first course. An extreme case might be an almost uniform first course, and a steadily increasing degree of taper for all the lower courses, with thicker ends at the bridge.

For the lower bass strings Dowland advised ‘Venice Catlines’. Mace said the same for fourth and fifth course basses and lower octave strings. ‘Pistloys’, which he considered the same type, were advised for the sixth and lower basses. The properties of both lower and upper strings were given as clear, smooth, well twisted, strong, and flexurally or ‘finger’ stiff. There is no mention of roping but if this were smooth, as strings are shown in paintings, it would look like simple twisting. Mace said that strings were stretched by at least one or two inches, which would be very large if he included basses. From the context he might just have meant the amount wound on a peg, but see later. The Mary Burwell lute tutor, of about 1670, says that good strings came from Rome, which had become the main producer after Munich. Lower strings also came from Lyons, but Mace thought they were variable. None of this is inconsistent with the fourth and lower fretted basses being tapered and continuing a scheme similar to Capirola’s instructions. If it has been assumed unconsciously that uniform strings were the obvious standard, then the requirement for some real evidence may lie here rather than with tapered strings.
6. Transition to the baroque lute

The seven course range of say D to g' is, perhaps surprisingly, the same as the baroque lute's C to f' over eleven courses, as used by the Gautiers. All the intermediate bass courses could therefore be added quite rapidly after 1600. As each course was added, the neck was tilted or angled consistently towards the bass side. This allowed a longer bridge to be placed symmetrically on the soundboard, using a narrow neck at the joint with the body (Ref 4). This was useful for converting highly prized older lutes. The angle made all the fretted courses down to say the eighth, as well as the lower basses, considerably shorter than the treble by about 3 to 6mm, and effectively rather more with the thicker loops at the bridge and acoustic end corrections. The nut and frets were perpendicular to the neck centre line, and significant angling of frets is unstable, without evidence, disliked by makers, and reduces freedom for other slight adjustments. These angled necks may be even more surprising than the angled bridges to modern players and makers. Coming across this detailed evidence in Ref 4 renewed my interest in the puzzle of shorter bass strings.

The physics of the extreme case (vi) shows that good tuning of these lutes can only be obtained with tapered fretted strings, where the thicker end is at the bridge for basses, and/or the thinner end for trebles. Compromised tuning for about a century with this design of lute is not likely. A great bonus of this solution is that tapering allows all directions of bridge angle, which also existed with the consistent neck angle, to be used with strings having otherwise similar properties. The intermediate courses would fit in with graded tapers and twist. This indicates that tapered strings may have been standard during all the golden ages of the lute. The early and late uses also imply continuity over the gap in the English sources.

The seven course lute might be seen as a peak of string requirements, but the baroque lute used the string elasticity even more cleverly. In detail, the older seventh course became three unfretted basses, which benefited from the symmetry but did not need special taper. The upper eight courses required only the taper of the previous six for tuning, and some extra taper for their shortening. If needed, some extra elasticity would improve the bass tone, which must have been worthy of all the extra courses. In comparison, modern lutes with no angles and uniform strings of equal lengths, but otherwise based on old designs, need more elasticity for good tuning.

A subtle advantage of taper is that octave strings can also be made accurate. With uniform strings the basses need high elasticity, since lengthening them might spoil neighbouring normal octave strings. Angled bridges would be a compromise unless the octaves were actually made stiffer. This illustrates how taper could allow other types of fine-tuning for individual strings.

Later on, metal windings could be added, reducing the unwieldy thickness of many lower courses. The higher density reduces diameter and gives thinner cores and greater elasticity. Octave strings also reduced the bulkiness. Another two lower courses were added for the lute used by Weiss. The experiments of Mersenne, before correct Newtonian theory was possible, may be connected with the added metal.

After the lute's decline, guts were split, cut, processed and polished to a cylinder, thus losing any trace of a long-range taper, as explained next in section 7. These, and other uniform fretted strings used on modern guitars and lutes, need to be highly elastic and equal in length, or with the basses slightly longer. Unfretted violins, harps, ouds, etc would be less affected. The difficulty of knowing a correct history, even in these later stages, should be a warning against rigid ideas.
7. Tapered guts
In the earlier paper 2 it was realized that tapering could be an intrinsic, natural, probably structural, shape of whole guts from various types of sheep. During this later work I found that a simple geometric estimate shows how the taper of guts could be exactly in the range of acoustic interest. For example, a typical 15metre long gut might give an average taper of order 2% on a string 60cm long: \((60 \div 1500) \times 100\%\). This can be halved for a tube collapsed into a solid string. Many different tapers could be produced by design, or as statistical variations. Producing a uniform string, in the modern sense described in section 2, would be improbable or difficult for an ancient string maker. Present day specialists, using whole guts, strive to make the expected uniform strings, and print a well-defined number on the envelope.

The ancient makers would have first noted the taper on the entire gut taken from an animal. This would have been correlated with the variations of tuning and fretting. Old, large-scale producers would not have discarded all their saleable, useable, and preferable tapered strings, and kept only the few uniform strings. It is important to remember that general tapered strings include the small class of uniform strings.

It is well known that twisting whole guts is a principal method for increasing the elasticity of strings. Twisting is just the ancient process of spinning many types of natural fibre. A further effect of twisting is to increase the degree of taper, as the string length is reduced. These effects, and their combination, may have been important discoveries for reducing stretch sharpening, perhaps even before Capito, when five and six course lutes were invented.

Tapered gut strings might be called 'conochords': literally conical gut lyre strings. Interestingly, an ancient Greek writer recommended strings that are even in thickness, and then refers in detail to freedom from defects, knots and loose fibres, but there could be no concept of modern precision and long-range uniformity.

Players and luthiers would have learnt how to select strings from large bundles or boxes, from different makers, travellers and markets. They would also master the skills of fitting and tuning, which would be normal practice. Modern orchestral woodwind players still need to make and modify their own reeds, and not simply choose them. Any excessive demand for a few scarce types of string might be relieved by the varied requirements of lutes with differently angled necks and bridges.

Capitola effectively used the lute as a detector of taper, a conometer, and the principle could be a guide in making the M strings. He advised noting the thickness of the three bass strings, and then matching the others. He suggested reversing strings to find improvements for the small differences of taper needed for the three upper courses. His uniform reference point, near the fourth course, had opposite tapers on either side, which avoids making a wider range of tapers. However, thin strings made of only a few guts would not easily give graded tapers, and he mentions these wider variations of taper on ordinary and 'da ganzer' strings, in contrast with M strings. For example, two guts can be twisted together in only two basic ways, to cancel or retain roughly the initial degrees of taper. Dowland mentions two strands at the end of a treble string, one thick and the other thin. This indicates a low chance of identical strands, rather than a proof of taper. The taper of thick strings containing many guts would be much more controllable, as noted in paper 2. A sensitive part of fitting strings would be finding two sufficiently similar natural products for an upper double course. The old tests for detecting and discarding false strings were equally concerned with matching slight falsenesses.

There are apparent inconsistencies between flexural stiffness and the high degree of bending in making up a hank of gut strings. These were flattened coils, as used for packaging rope or modern silks. Between the two end loops, the central part was kept tightly bound, according to Mace. The strings were up to four times the length needed on a lute. Six long bass strings could give a substantial hank about 20cm long and 1½ cm thick. Eight-dozen trebles would give the
same substantial size of hank (another factor of 16). This length for a hank may be indicated by an engraving of a rig for winding finished strings on (Ref 5). The lute books say that even treble strings were well twisted, unlike modern trebles. This appears to reflect ancient but incorrect ideas of strength (Ref 5), possibly related to the spinning of other fibres. This twisting, and also chemical processing, may have given flexibility for bundling. Stiffness might have been stressed in the English sources as relative to weak rotting strings. Thin twisted strings may have tended to self-destroy when tensioned on a lute, to form the flaccid, curly, broken strings seen in paintings. Dowland warned of curliness on poorly twisted strings, even before use. If this conjecture is correct, old gut for trebles could have needed very high intrinsic strength.

Taper may indicate how strings needed to be drawn out of a bundle. The early bundles of Capirola had one end thicker than the other and were presumably long. A thick end of a string would be pulled out. For the later coiled hanks of Gerle, Le Roy and Mace, a thin end might be worked back carefully to an end loop. This operation would be repeated until a length could be tested for falseness, by plucking between the hands, before fitting on a lute. Many of the old lute books have a schematic diagram of this test. This discussion of bundles and hanks has tried to fit together into a coherent picture the many cryptic phrases in the sources.

8. Possible future for tapered strings

Until a few decades ago, tapered gut was in fact used for a special fishing line. A length of about 4 metres with a large taper of order 50% connected a small hook to a thick line. Just as this product was declining, the lute revival was starting with the new nylon replacement for fishing lines. It is not easy to see how a synthetic material could provide tiny degrees of taper on the short lengths required for strings. A rough attempt might wipe a monofilament such as nylon with an abrasive powder, starting at successive positions increasingly further from an end. It is possible that players have noticed some effects of irregularities on strings, but the regions of wear during playing would not give an effective taper, and are similar across all strings.

Modern makers of strings may like to try some tapered whole gut constructions. The correct, species, age, etc of the source will be important. Players and makers could learn how to use a good batch, with help from Capirola and the present analysis. Many tuning details, and maybe effects on temperaments, can be expected from this highly versatile mechanism, with implications for playing techniques and the music. Modern technology and economics might want to treat a tapered string as an even more highly specified item. There would probably be no demand for a return to large-scale manufacture of a variable natural product, but there might be a developing country with a surviving tradition. If this is not possible there is still interest in seeing how some baffling features of old lutes may have produced excellent results with original strings, but not with modern strings. In particular, all of the old different bridge angles, and neck angles, can now be accommodated by taper. Now one may inquire why these variants were made, rather than how on earth they were tuned. In terms of soundboard geometry, the effects of these angles initially seem rather small. Visual appearance is simple in principle, but may not be easy to separate from any acoustic intentions. A significant acoustic effect of slightly angled bars and bridges was found later in the work on soundboards in paper 6. It may become regrettable that some old lutes have probably been damaged by modifications to allow the use of later strings.

9. More elasticity

There is still some use in finding greater elasticity for bass strings, in order to improve all the effects in section 1. It has often been noted that gut trebles can give a feel in phrasing. Conversely, extra security may be useful in the bass and this would require more elasticity. Modern synthetic strings have very stable trebles and less secure basses, which is the opposite trend. For Capirola’s tapered bass strings with lower elasticity, a more detached phrasing may
have been necessary. In addition, old twisted trebles would be more secure, so the initial surmise may seem musical, but not historical. These effects are not a simple reflection of good tuning for tapered strings. In paper 2 on angled bridges, musical phrasing had seemed to be the sole definite advantage from angled bridges with more elastic basses.

In relation to any higher elasticity, as indicated by Mace, there is an infernal possibility, known to string makers but not used in earnest for a century (eg Ref 5). Twisted strings were stored in an atmosphere of burning sulphur for up to a week. The initial purpose may have been to inhibit rotting. I recall dimly, from some involvement with molecular biology, that sulphur was said to break down strong hydrogen bonds between the various levels of helical structure in collagen, which gives high strength and toughness to tendon, skin and bone. This might retain sufficient strength for bass strings, while giving greater elasticity. It appears different from the vulcanization of rubber, which cross-links a useless liquid to form a solid with great elasticity, but limited strength. A short process was last used on thin violin strings, where it was finally considered unnecessary. A long process, which would be needed for thick basses, may have been overlooked. Perhaps some string makers would be willing to swap their micrometers for gas masks. A long initial cleaning with natural potash, and final soaking in olive oil, might also be important processes (eg Ref 5).

10. A conochordal coda
It is an odd thought that any surviving samples of old strings probably would not have led to a rediscovery of taper, and might have impeded it. Some real evidence is grazing in fields today, and the remarkable properties of their innards have assisted this survival. This is just one of four types of interconnected evidence: physics, biology, Capirola et al, angled necks and bridges, with Ockham’s razor as a possible fifth.

Detailed analysis has been necessary to explain features that are puzzling to our modern expectations. A final, more relaxed position could be a realization that the old gut products had many natural variations, which the ancients learned to live with, adapt and perfect, over many centuries. It might be a modern mistake to think in terms of rigidly applied mechanisms.

Some differences between old instruments and their modern counterparts are a matter of degree, and reasonably clear. Others are more fundamental, as in the imprecise handmade bore of brass instruments, which broadens resonances and allows flexible tuning; and now the complexity of naturally tapered and uneven gut lute strings. Although the effect with trumpets was mentioned as early as paper 2, it appeared to be no help during the work and only later was the comparison recognized. (Guts are sometimes likened to trumpets, but ‘salpingochord’ seemed confusing.)
The present exertions over half an inch of gut, and some details of tuning, appear to have wider significance for the history and design of lutes. The results are also relevant to all other old fretted gut strings for larger lutes, guitars, vihuelas, viols, etc, and perhaps for instruments of different cultures.

References
1. Vincenzo Capirola’s Lute Book. 1517.
   some excerpts are in paper 2 but the full text would be useful in reading section 4.
3. Thomas Mace: Musick’s Monument. 1676.
I am also grateful to George Stoppani for some support on my generalizations about gut geometry, and to several Lute Society members for some reassurance on the details of angles in old lutes.
ACOUSTICS OF LUTE SOUNDBOARDS AND BARS (4)

1. Introduction
Lute strings are discussed extensively, but little is printed on the acoustics of lutes themselves. The mechanics of musical instruments is intriguing, particularly concerning lost beautiful sounds, but notoriously difficult to extend beyond a few simple principles. Recently, I have found that much of the resonant behaviour of a soundboard might be understood, and with just one relation used in various forms. This may be of most interest to scientific researchers of instruments; and a complementary view for makers, whose designs nowadays are based on old lutes in a nearly original form.

The soundboard, or belly, is a vibrating elastic plate, but with many complications such as different degrees of stiffness across and along the wood grain; variations of thickness; an oval outline; several braces or bars attached underneath; the bridge and the rose hole. The scientific and practical challenge is to find reasonable simplifications that will enable some useful analysis.

2. Soundboard without bars
The starting point here is an elastic beam, familiar from a ruler on a desk edge, or a reed in an organ. The allowed or natural frequencies of vibration in this one-dimensional problem are predicted to be

$$f = \left( \frac{s}{l^2} \right) \left( \frac{1}{4\pi\sqrt{3}} \right) m^2 \sqrt{\frac{E}{\rho}}$$

Here s is the beam thickness, l is the vibrating length, E the stiffness or elastic modulus, and \( \rho \) the density of the beam. This is a standard result of solving a fourth order partial differential equation expressing the balance of elastic bending and inertial forces (eg Ref 1). A tensile modulus appears due to stretching on the outer surface of a bend, with an opposite compression inside. Resonance occurs when an oscillating driving force, such as the end of a string on the bridge, has a frequency equal to one of these natural frequencies. The amount, or amplitude, of vibration is large at a resonance peak but falls off for other driving frequencies. The factor m has a series of allowed values, which depend on the conditions at the ends of the beam. For ideal vibrating strings the analogous values form a neat series of perfect harmonics, but the series of m does not generally give pleasant sounding overtones. For example, the ruler has one end clamped and the other free, while for both ends clamped m and f have the highest series of values. Freely hinged ends give the lowest series for held ends, and more harmonic values for f, with m equal to \( \pi \), 2\( \pi \), 3\( \pi \), etc, or n\( \pi \). The next stage explores how the frequency relation and some idealized end conditions may be applied to a lute belly.

Firstly, vibrations with bending across the grain of a rectangular soundboard, with free ends and no bars, involve a low transverse stiffness \( E_t \) with a value of about 0.5GPa (or 32tons/sq inch), from data books and a few tests of my own on soundboard pine. These transverse vibrations are predicted by the equation to give our lowest frequency \( f_t = 9.2\text{Hz} \) (or cycles/second) for a typical soundboard thickness s of 2mm, a width l of 30cm, and a density \( \rho \) of 0.6gm/cc. This is for two hinged sides, and for clamped sides \( f_t = 21\text{Hz} \). The precise numbers should not be taken too seriously since this is really a semi-quantitative 'thought experiment'. Secondly, longitudinal vibrations with bending along the grain, like the ruler, involve a much stiffer \( E_l \) of about 14GPa and their frequency is \( f_l = 22\text{Hz} \) for a typical larger body length of 45cm with hinged ends, and for clamped ends \( f_l = 50\text{Hz} \). These values are for the first mode, or
lowest value of $m$, with higher frequencies for higher modes. For example, the next $f_i$ values after 22Hz are 88Hz (x4), 198Hz (x9), etc.

For a plate with all the edges held, as for a lute belly, any transverse mode must be combined with any longitudinal mode to give a resultant $f = \sqrt{(f_t^2 + f_l^2)}$, which gives an even less harmonic series. It is therefore seen that for this geometry the lowest resonance is greater than the basic $f_i$, with a succession of higher modes produced by the $f_i$. Above $f_i$ this is similar to the harmonics of a string, or the available notes on a natural trumpet, except that the spacing increases more due to the dependence on $m^2$, rather than just $n$.

On a G lute the sixth course G is at 98Hz, and the 12th fret g' on the first course is at 784Hz, so that there may appear to be several low resonances able to assist the string vibrations in forcing the soundboard. However, the presence of bars raises a very serious problem since they would prevent the low transverse modes found above, although not greatly affect the longitudinal vibrations. Alternatively, if longitudinal vibrations were considered for the small lengths of belly between the bars, their $f_t$ would be greatly increased as $1/l^2$ by an enormous factor of order 100. These very high $f_t$ could, however, be combined with the very low transverse $f_i$. In contrast, tests on plates and bellies showing patterns of stationary points, or nodes, revealed by sprinkled powder when vibrated by a loudspeaker, actually gave more general states with moderate frequencies (Ref 2, with relevant details described below). Consequently, I had thought for three years that nothing predictive or theoretical could be said, until the heat wave of July 2006 with house painters outside required a long but interruptible occupation indoors.

3. A new treatment for bars

A solution that eventually emerged is to adapt the basic equation to the bars themselves, and so find a completely new transverse stiffness, effective mass, and frequency $f_t$ for the whole belly. This is then combined with slightly modified longitudinal frequencies $f_i$. For a bar vibrating when attached to the belly, the beam thickness or depth is say b, analogous to the soundboard thickness s. A typical value for b is 15mm, which by the equation above increases $f_t$ by a large factor of 7.5. Also $E_t$ must be replaced by $E_i$, since the grain of the bars now lies across the belly, giving a further factor of 5.3, and a total of 40. This increase in $f_t$ is slightly reduced by an increased effective density, since for a spacing between bars of typically 50mm the mass of this extra length of soundboard $w_s$ is added to the bar with a width $w_b$ of about 4mm. The new effective density of a bar is

$$\rho(1 + sw_b/w_s) = \rho(1 + 2x50/15x4) = 2.67\rho$$

which reduces $f_t$ by a factor of 1.63. The transverse frequencies are therefore increased by an overall factor of 24, giving a new $f_t = 220Hz$ (9.2x24) for the lowest mode with hinged sides. Clamped sides give 500Hz, but are a less likely condition, because of the deeply scalloped ends of the bars. The soundboard contributes negligible transverse stiffness and acts mainly as a connecting sheet, which is also vibrated or flapped to make sound waves in the air.

The transverse bars can scarcely affect the longitudinal stiffness of the belly, but will add mass to the soundboard and increase its effective density, now by a factor of

$$1 + bw_b/w_s = 1.6$$

This reduces the previous $f_t$ by a factor of 1.26, giving the frequency of the first longitudinal mode as $f_l = 17Hz$ (22/1.26) for hinged ends, or 40Hz with clamped ends. The condition at the tail end may be closer to a hinge. At the neck the condition may be more like a clamp, but this region can be expected to be less active. The old makers have simplified this analysis, and probably their own work, by using soundboard wood for the bars, which therefore have the same values of $E$ and $\rho$. However, general equations for the resonant frequencies can be written as:

$$f_t = (b/l_b^2)(1/4\pi\sqrt3) m^2 \sqrt{(E_{bt}/\rho_b(1 + sw_b\rho_b/bw_bp_b))}$$

$$f_l = (s/l^2)(1/4\pi\sqrt3) m^2 \sqrt{(E_{sl}/\rho_s(1 + bw_b\rho_b/sw_s\rho_s))}$$
The subscripts s and b refer to soundboard and bar properties, \( l_b \) is the bar length or lute width, and other symbols are defined above. A bar width \( w_b \) is preferable to a more consistent but confusing 'length', and hence \( w_s \) is used for the soundboard length between bars. These mechanisms may seem an odd mixture of the obvious, with perfect hindsight, and the counterintuitive. The various contributions can also be seen clearly from the differential equations. An even more general expression can account for stiffness in both soundboard and bars, but this is not necessary here. The relative elastic and inertial forces will differ for the various bars on a lute, increasing at the neck end where \( l \) is smaller, but where vibrations may be weak due to a greater distance from the bridge, and the intervening rose. Also, the active lower belly is far more rectangular than may be suggested by the overall shape of a lute.

The lowest resonances of the lute belly can now be formed from the new much higher transverse \( f_t = 220\text{Hz} \) in combination with the new slightly lower longitudinal

\[
\begin{align*}
    f_l &= 17, 68, 153, 272, 425, 612, 833, \ldots, \text{Hz} \\
    f &= 221, 230, 268, 350, 612, 862, \ldots, \text{Hz}
\end{align*}
\]

Higher transverse modes then occur at \( f_t = 880, 1980, \ldots \text{Hz} \) for hinged sides, and would be combined similarly with the longitudinal vibrations. The lowest resonance at 221Hz has a nodal line around the edge of the belly attached to the body, and the centre oscillates in and out. The second resonance at 230Hz has an extra nodal line across the belly, with opposite motions on each end; the third resonance has two such transverse nodes, and so on. The second transverse mode has a nodal line down the centre of the belly, so the ends of a bridge could have opposite motions. The modes are shown in the later diagram.

This theory appears to explain reasonably well the experimental patterns, which could only be obtained for four of the lowest resonances in Ref 2. The test frequencies can even be used to calculate effective values of \( f_t \) and \( f_l \). These are about 30% higher than above, but a test frame with rigid sides was used, and also a complete lute that probably had a heavier belly thirty years ago. The theoretical end condition of perfect hinges is probably too relaxed, and some effect must be expected from the approximations in geometry and material properties. The wood in quarter-sawn bars would have a different plane of bending from that in the soundboard. Clearly, the calculated frequencies and shapes of modes cannot be expected to be exact. The main purpose of this difficult search has been a general prediction of the moderately low resonances purely in terms of the dimensions and physical material properties of a lute.

4. Bars and nodes

In both theory and tests, the nodes in low resonances do not generally occur near bars, which just provide an overall elasticity and mass of belly. In order for bars with a typical spacing of 5cm to behave as hinged ends and nodal lines, the equation requires a high longitudinal frequency \( f_l = 1782\text{Hz} \) (or \( 22\times(45/5)^2 \)). A larger spacing of 10cm with fewer bars would need \( f_l = 445\text{Hz} \) (\( x(5/10)^2 \)). These much higher modes for separate panels of soundboard could then combine, as suggested initially in section 2, with the first low transverse \( f_t = 9.2, 37, 83, \ldots, \text{Hz} \), to give many very closely spaced higher frequency resonances. Therefore, by say 2000Hz, three or four systems of resonances would be appearing, based on the first then second 'overall' transverse modes; the first longitudinal mode for panels; and then the third transverse mode, and so on. The later detailed lists of frequencies illustrate this, and label the modes, but are not meant to imply great certainty.

5. The rose and air resonances

The lowest resonance found above is about 220Hz, or an a below middle c', so that there is still a significant problem of how the lower notes radiate from a lute. Another mechanism involves the well-known Helmholtz resonance, where air inside the lute body can act as a spring and a
region of air at the rose is the oscillating mass, but with no physical possibility of higher modes. In practice, G lutes have this resonance $f_b$ near 130 Hz, or an octave below middle c'. In general $f_b$ varies as $\sqrt{V/(A/nV)}$ where $V$ is the enclosed volume of air, and the neck has a cross section area $A$ and a length $n$. This gives good values for say a Bordeaux bottle with a well-defined neck and an $f_b$ also close to 130 Hz. However, I have found that $\sqrt{2d/l_b^3}$ is a better guide for a rose of diameter $d$ with a typical carved design, and a lute body with a width $l_b$.

A further type of resonance in the enclosed air can arise from standing waves, similar to an organ pipe, but a maximum half wavelength of order 45 cm gives a frequency above 370 Hz, or 550 Hz for transverse waves, with a series of higher modes. For this reason, the low pitch of many string instruments relative to their size requires the operation of soundboard and Helmholtz resonances. However, a single low resonance, as for an organ pipe or selected on a wind instrument, would not be useful. For example, players know that a fifth course c near the Helmholtz value may be annoyingly prominent. Also, a non-harmonic series of resonances might be expected to be better for projecting all the different low notes of a lute.

The lower resonances predicted for a lute are now say: c(130 Hz), a(221), a'(230), c'(268), f'(350), f''(370), b'(479) etc, whereas the range of the open strings on a seven course G lute is D(73), G(98), c(131), f(175), a(220), d'(294) and g'(392).

6. Neck and body

Two further possibilities can be suggested for the lowest octave of a lute. Firstly, a slight bending of the entire lute might be treated very roughly as a beam of length 70 cm with a thickness of 10 mm, from an average for a neck depth of 15 mm and a combined thickness of body and belly of 4 mm. This ‘beam’ has two free ends strongly driven at the bridge and nut, and the initial equation gives a lowest mode of about 70 Hz. The wide fingerboard of a lute may increase the effectiveness. This source of resonance may be particularly important for lutes with many bass strings, and for long-necked types.

The other possible resonating element is the body shell, where analysis is far more complex than for a flat plate. However, the thickness is similar to a belly, there is less difference between transverse and longitudinal moduli, and there are no bars. Hence the lowest frequency may be of order the initial $f_b$ = 20 Hz for a belly, combined with a similar value for $f_b$, giving a lowest resultant of 28 Hz. Players can feel these deep, pleasant vibrations, which may be more important for the lute than for other string instruments of a similar size but a higher pitch.

7. The overall response

In a complete lute all these elements: the soundboard, bars, enclosed air, and more weakly the neck and body, are coupled together and function co-operatively. This could shift and broaden the strong resonance peaks for the belly and air; extend the response to lower frequencies for the neck and shell; and partially fill gaps between all the separate resonance peaks. Non-uniform dimensions, similar to the bores of old trumpets, and damping effects may also broaden the peaks. Some players have noticed that tuning a lute to even a slightly different pitch can change the entire sound and feel, and this could result from a different use of the fixed spectrum of resonances of the instrument. Players will also be concerned with how resonances combine in chords, and change while playing a melodic line. Transitions or jumps between very different modes, perhaps even for small common intervals, could greatly affect the feel or ease of 'speaking'. Particularly at high frequencies there can be several possibly superposed modes with the same pitch, known as ‘degeneracy’. Some basic modes can also combine together to form new modes.

8. Comparisons with violins and guitars

There has been much testing of violins (see Ref 2 and section 15). The Helmholtz resonance is near 290 Hz or the d' string, and the first top plate resonance is near 440 Hz or the a string. Unlike the lute, the bottom plate is similar to the belly and strongly coupled by the sound post.
Violinmakers tap their plates and flex them by hand. This is known to assess the low resonances, whose correct placing is crucial, and almost sufficient to ensure the higher modes and a good instrument. The method is needed because material properties are not consistent and have to be compensated by small changes during construction. Has modern lute making adopted or reinvented similar methods, did any ancient writers leave descriptions, or did luthiers adopt such methods later for violins? Perhaps the three-dimensional blocks for violin plates are more variable, while a good stock of thin pine is consistent enough to use less variable dimensions. Similarly, the present calculations for uniform separate components of a lute belly would not be possible for the complex contours of a violin plate. In this respect, in addition to tuning schemes, and elegant intersections of curved and flat surfaces, the lute appears to be a rather scientific artefact, reflecting its Middle Eastern origins.

In contrast, for a modern guitar the fanned bars will give an increase in longitudinal stiffness, completely unlike a lute, but also some increase in transverse stiffness with an added mass. This may give a few basic low longitudinal modes, each combined with several lower transverse modes, which is the reverse of a lute. The fanning would inhibit bars from acting as nodes, and any resonance of panels would tend to have the same pattern as the basic modes. Hence the greatly different types of mode and the many close high resonances of a lute would be absent in a guitar. This might imply that a guitar belly could be more isotropic, roughly as for plywood or a plastic, whereas the predicted lute resonances depend strongly on the large difference in moduli. The very slightly angled bars of some old lutes would change the relative moduli and masses, but it is uncertain whether this was a deliberate fine adjustment.

9. The bridge
The bridge on a lute must be important, since it is an added mass and also the region through which the strings drive the belly. The extra mass of a bridge m_b relative to soundboard mass m_s can be shown to decrease the resonant frequencies by a factor of order

\[ \frac{1}{\sqrt{1 + 2m_b/m_s}} \]

Complete solutions for this type of system are more complex than for an unloaded, unforced beam or plate. For a typical bridge this factor may vary from about 0.95 for a whole belly, to 0.85 for a single panel with a length w_s of 10 cm, and 0.75 for a 5cm panel. The tests in Ref 2 showed a 10% decrease in the low resonances, but used a distributed forcing by a loudspeaker. A slimmer, lighter bridge will generally reduce frequencies less, and can deliver a greater proportion of the energy of a string to the soundboard. There might also be a weak trade-off or optimization, since a longer bridge, as well as increasing the mass, will also reduce the width of free soundboard at each end or the bridge, which will increase stiffness and frequency.

The motion of the bridge means that longitudinal modes which would have nodal lines close to the antinodal bridge will be weak or absent. A bridge at say 10cm from the base may therefore have weak fourth and fifth modes, at 350 and 479Hz in section 3, which would have nodes at 11 and 9cm from the base. Multiples of these modes would also be weak. This could leave a gap in the spectrum of resonances, perhaps needing compromises in positioning the bridge and bars. A bridge is not usually placed too close to a main bar, which would be given an effective increase of stiffness and mass by the bridge. This could still allow modified low transverse modes. In contrast, for the high resonances of panels a main bar acts as a node so that a nearby bridge would provide little forcing.

10. Forcing by strings
The force exerted on the belly by a vibrating string can be estimated. An important initial point is that a string has a slowly decaying amplitude after being plucked, whereas the soundboard only has a forced vibration which ceases when a string is damped. Various contrasts are bowed strings and ringing percussion. For a string at a tension T of say 3kg, plucked downward by an amount p of 5mm, at a distance q from the bridge of 15cm, the force perpendicular to the belly...
has an initial amplitude of $pT/q$, which is 100gm or 4oz. This force can be calculated, from the soundboard theory, to produce a deflection of order 0.04mm or 40µm in our typical soundboard, which can be felt but not seen. In detail, this triangular wave reflects back and forth from each end of the string, producing pulses of force on the bridge and soundboard.

Vibrations of a string with general initial conditions can be represented usefully by a sum of harmonics, and the amount of each may be calculated by Fourier analysis. Harmonics with nodes at the plucking point will be absent. This is particularly noticeable for a lute in the lack of all even harmonics for plucking at the midpoint, which for a fretted note is usefully nearer to the rose. Another feature is a decreasing proportion of higher harmonics $n$. This is more pronounced for plucking nearer the midpoint than for a position nearer the bridge, which gives a brighter sound. A similar effect occurs with the method of plucking. For a slow displacement the amplitudes decrease as $1/n^2$, whereas for a striking action the dependence is $1/n$, giving more brightness. This is a summary of the type of detailed theory found in maths texts such as Ref 1.

An important analogous effect will also occur in a freely vibrating soundboard, so that while there are many higher modes they are progressively less strong. For a simple initial displacement of a beam or plate, the amplitudes decrease as $1/m^3$, and for an impulse as $1/m^2$. In contrast, for a plate forced by a string, the amplitude near a resonance varies as $1/(f_m^2 - f^2)$, where the resonant frequencies $f_m$ themselves depend on $m^2$, and the values of $f$ are the driving frequencies in a string. Various types of damping prevent infinite amplitudes and also slightly reduce the $f_m$. A soundboard may therefore have some strong higher resonances. The amplitudes will also be affected by the position of the driving bridge, as in section 9.

A plucking action parallel to the belly still gives an oscillating force along the length of a string, which will tend to tilt the bridge and so lift the soundboard. This is weaker than a downward action, but not the simple expectation of a zero effect. Different plucking directions, and also strings with different positions on the bridge may excite different modes in the belly. The motions of strings may couple together. There will also be important transients at the beginning of a note, and real strings may have slightly inharmonic components. Both of these details are known to affect greatly the perceived sound, which could reduce the value of elaborate standard analyses.

11. Treble and bass bars

The small angled treble bars between a bridge and the body side could stiffen the belly and raise the frequency of all types of mode in an area without other bars. The larger bass bar runs across the grain without meeting the bridge, and would raise any localized low frequency transverse vibrations of the large lower panel, which lacks other bars and also has a frequency lowered by the bridge mass. The bars may also reduce excessive amplitudes close to the forcing bridge. These minor bars might even be viewed as an extension of the bridge to form an effective bar at the wide end of the belly. The scalloped ends of a bridge would give a gradual transition to the smaller bars and belly.

12. Discussion

These ideas could be developed at great length, but some main points are worthwhile.

(i) Soundboard thickness $s$ is clearly important. It is sometimes inferred that thick soundboards give less treble resonance, but theory shows that frequency increases with $s$, because elastic forces increase more than mass. A thick belly or many large bars on a heavy lute require more effort to drive so that the low Helmholtz resonance may predominate. Also, a thinner soundboard will be easier to force and tend to produce higher harmonics more easily. The same balance of effects will also apply to bar depth $b$. A matching or optimal combination of lute, strings and player is needed, and physics may analyse this.

(ii) The size $l$ of real lutes generally varies inversely with pitch. This means that if thickness $s$, depth $b$ of the bars, or rose diameter $d$, are increased with $l$, the resonances will have the same
relative position in a lute's range. However, if s, b, or d is kept constant, the resonances can be made relatively lower for larger lutes, which may increase their sonority. (iii) The scallops on bar ends may be significant in behaving more like a hinge than a rigid clamp. Their detailed shape and joint with the body would affect the resonances. Adjusting the thickness of the edges of a belly may be similarly sensitive.

(iv) The depth b of a bar is predicted to be acoustically important, and more so than its width w, or the spacing w, from the equations for f, and f, The new treatment of bars provides a rich and testable interplay of elasticity and mass in determining many different resonant modes. One example is that a larger spacing of fewer bars, in addition to the large lowering of the resonances for separate panels, where f1 was reduced from 1782 to 445Hz, would also decrease the frequency of the lower transverse modes. Compared with the original bar spacing w, of 5cm, a spacing of 10cm increases the effective density for the transverse mode at 220Hz, so lowering f1 to 173Hz. There is also a decrease of density for the longitudinal mode at 17Hz, raising f1 to 19Hz. This gives a new lowest belly resonance of 174Hz, compared with 221Hz for the 5cm bar spacing, as seen in the lists of resonances. Such a light early lute might not yet have acquired a bass bar, and a Lute Society drawing by Stephen Barber, based on original lutes, has just this pattern of bars. A potential problem might be a belly resonance unnecessarily close to the Helmholtz value, and there could be a reduced strength under the tension of the strings. There is usually a mixture of bar spacings, which may result mainly in a greater variety of higher modes.

(v) Structural stability is an important, and primary, purpose of bars and it will increase with depth b, width w, and number of bars or 1/w. Hence it may be possible to increase bar width, while reducing depth and/or number of bars in order to lower the resonances. Squarer sectioned bars have been reported for older lutes (Ref 3).

(vi) The dominant effect of bars in determining the lower resonances suggests that the high quality old pine favoured by makers is required mainly for the higher resonances of separate panels. Tests on violins have indicated that the finest old instruments have strong high resonances.

(vii) In principle, equations such as those predicting f, and f1, for both the low overall modes and the high modes of panels, could be used to investigate and suggest the geometry and properties needed for different spectra of resonances.

The present explanations, arising from attempts at full understanding, are the type of insight that can be provided by physics, rather than detailed plans, or a recipe for the perfect instrument.

13. Possible use for experiments

Although this research grew out of sheer curiosity, it might be used practically for designing some necessary experiments, or as a method for repeatable lute making, and perhaps modified or optimized designs. The physics simply tries to predict what happens in an instrument during playing. How it sounds also depends on the sound radiated in air; an enclosing room; and finally on perception, involving physiology, neuroscience and artistic taste. Qualitative descriptions of sound quality have been avoided here, but may come to mind and might even be a good guide without details of rooms, ears and brains. The predictions for bars are a possible example.

For repeatable lute making, one could use powder or optical patterns in the same way as tapping and flexing. Lute parts could be tested during construction or a restoration, and the best final results copied. It would be necessary to test the reliability of such a method. An advantage of this use is that it avoids any need to understand the later sound radiation and perception. Such tests could also be planned to provide useful basic experimental data. Although turning a dial to find a resonance is not difficult, it would be important to know how to support a plate and plan a truly meaningful series of tests.

If the aim is to modify designs, then systematic variations based on any idea can be tested purely empirically, but physical understanding would be more beneficial, if one has the inclination and time. At the start of the early music revival, there was considerable interest in technical testing,
often involving substantial modern alterations of the ancient designs. Ref 2 may be a sole surviving example of this approach. Now that original instruments are followed more closely, the emphasis seems to be on the finer details and craftsmanship, rather than the larger scale mechanical principles. Makers of modern violins, and some other instruments, are remarkably open to scientific testing. The enormous scope for personal choice in the precise resonance patterns and details of the best instruments may produce little fear for trade secrets.

14. Early guitars and vihuelas

After the work on lutes, I learned that early guitars and vihuelas had only two bars, immediately above and below the rose. The result is a large lower area of free belly, fitted with only a bridge and attached to the sides. This would be expected to give resonances based on low longitudinal modes with a length $l$ of say 30cm, which for hinged ends are

$$f_l = 50, 200, 450, 800, 1250 \ldots \text{Hz}$$

These can be combined with low transverse modes, as found for a lute at $f_t = 9.2, 37, 83, \ldots \text{Hz}$. The many resultant resonances are given in the lists, showing successive appearances of a new $f_l$, with a group of closely spaced resonances from the low $f_t$.

With some irony, this is the simple type of behaviour that could not be applied to a lute in the initial work. The patterns of modes discussed qualitatively in section 8 for a modern guitar with fanned bars are similar, and therefore continue the character of the early guitar. There are some very low resonances of 51, 62 and 97Hz, which are well below the belly resonances of a lute. The lowest are not even needed by the guitar's strings, and this is very different to the investigation of a lute. The roses of early guitars were very elaborate, probably giving a low Helmholtz resonance and also a weak projection of sound, whereas modern sound holes are large and open. The uniform thickness of about 10cm for a guitar body will produce standing wave resonances in the enclosed air based on about 1600Hz, combined with lower modes like those for the lute, which may give some general brightness. Apart from the difference in elastic moduli, giving close groups of resonances, the modes of early guitars may be similar to those of a skin membrane with negligible stiffness, as in a sarod. The broad impression that a guitar has strong low and weaker high resonances, compared with the opposite character of a lute with weak low and stronger high resonances, is supported and explained well by this analysis.

The effect of a bridge can be examined as for the lute. All the vibrations involve the whole belly, and will be reduced in frequency by about 10%. The position of the bridge relative to nodal lines of the longitudinal modes will produce gaps in the spectrum of resonances. For example, since the bridge is about a third of the distance from the base to the first bar, modes based on the $f_l$ of 450Hz in the list may be weak or absent. This is the region where a lute with fewer bars was predicted to have strengthened resonances, in sections 4 and 12. The length of a relatively rigid bridge may also weaken higher transverse modes. Lower modes may be stronger, as for the basic first mode, the second with a rocking motion, and the third with nodes near the ends of a bridge. This would be complex and depend also on the position of an active string. These effects would be different for a lute, which has a main bar close to the bridge.

Vihuelas can be expected to have similar resonances to guitars, and an interesting rare use of powder patterns is summarized in Ref 4. A consort of four differently pitched instruments was designed from a single historical example. Similar patterns were obtained for each vihuela, by removing wood from the bellies, but the frequencies of the resonances were relatively higher for the higher pitched smaller instruments. This appears to be an example of the discussion in section 12 of a belly thickness $s$ being relatively thicker for smaller instruments. From the available data on frequencies, the extra thickness of soundboard, relative to the original pitched in $F$, can be estimated as about 6% for the model in $G$, and 20% for the highest model in $C$. This indicates thicknesses almost midway between a fixed constant, and values scaled with string length and frequency. It is not known whether either of the two extremes, or the intermediate
value, would have been the ideal aim of an old maker. There will be a practical limitation on using much thinner pine for smaller instruments. This could lead to the use of relatively thinner soundboards on larger instruments, which can therefore have relatively lower resonances with greater sonority. Conversely, the smaller instruments could have a relatively brighter sound, perhaps with more specialized uses. Tests that show large effects, as here, are a good example of where physics may provide criteria and help the design of experiments.

It may be a surprise that these ideas for treating soundboards appear to be new. This could be a consequence of only a few people being interested in the easiest case of the early guitar. In contrast, the highly popular violin seems intractable. Having seen how to treat the difficulties of bars for a lute, the more complicated modern guitar could be attempted, and perhaps some very limited application to a violin. However, the present approach devised for a lute would not have seemed appropriate solely for these more difficult cases. The banjo may have escaped attention, and could be interesting for basic theory and experiment, with a membrane, bridge and strings.

15. Footnote
After this work, a search of the literature found a large recent academic book (Ref 5). This treats most western instruments and reviews mainly experimental studies. There is a half page summary of Ref 2, but nothing more on the lute; much data on the separate plates of violins and guitars, but no theoretical predictions; and interesting aspects of coupling between component parts. Strings developed for modern instruments do not present the mysterious difficulties of old strings. Surprisingly, there is no discussion of computer methods for solving the differential equations. This could be interesting for showing detailed modes, but lengthy and with several internal problems. Also, each solution can only refer to just a single practical combination of lute geometry, string, note, plucking method, etc. This could put any general understanding, or a spectrum of resonances, far from reach. However, there must be a good chance that in the last ten years a suitable program or code has tackled some instrument. This book also confirms that distinguishing good instruments from poor ones is still presently beyond scientific explanation, although it is not difficult to pick a good one and then do some tests. Finally, it is worth noting that understanding of instruments is at best semi-quantitative. This is a strong contrast with the high accuracy of tuning that is required by any player.

References

Key for table
Mode (1,2) is the first overall transverse combined with the second longitudinal, etc.
Mode (3,1p) is the third transverse combined with the first longitudinal for separate panels, etc.
For the 10cm bar spacing, note the many middle range resonances between 400 and 800Hz, in contrast with the 5cm spacing, but no lack of higher resonances.
For a guitar, note many lower resonances based on low longitudinal modes, and the same type of mode for higher resonances. A smaller typical body size of 25cm would raise all frequencies by a factor of 1.44 (or (30/25)^2).
The fifth column is 0.8 times the third to represent a lighter lute. The sixth column is an example of the effect of a bridge and treble bar on the lower panel. This is explained further in paper 6.
## Soundboard Resonances

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Note: Hz for 5 cm bar spacing, Hz for 10 cm bar spacing, Hz for 2p as 1p spacing for 5 cm.
Schematic diagrams of some soundboard modes

(1,1) +
(2,1) + -
(3,1) + - +

(1,2) -
(2,2) - +

(1,3) + - +

(1,1p) 5cm bar spacing
(4,1p) 10cm mixed spacings possible

5°
This communication has three parts: (1) an idea that the application of string tension might change the resonances predicted for an unstressed soundboard, then (2) a development of this analysis into a theory to understand the strength and stability of a lute, with practical predictions of bar sizes and positions. Part (3) uses some of this work to explain the concave surface of the ribs on many lutes. More background can be found at the beginning of paper 1.

1. STRING TENSION AND SOUNDBOARD VIBRATION

A while after the main paper 4 on soundboards and bars, a further possibility came to mind. The resonances found in theoretical work and tests on lutes without strings might be altered under the considerable tension of the strings. Previously, this may have been unforeseen or neglected, but makers might have become aware of some effect. For vibrating strings, their elastic bending gives a small, but inharmonic, frequency increase to the main component resulting from the tension. This suggested an analogous but opposite behaviour for the soundboard. In paper 4, the frequency of longitudinal resonances, for hinged ends, was

\[ f = \frac{s}{2} \left( \frac{\pi}{4\sqrt{3}} \right) n^2 \sqrt{\frac{E}{\rho}} \]

with an extra factor for the inertia of the bars. If one imagines a soundboard under a tension \( T \) and neglects its elasticity, a frequency of vibration could be defined as

\[ f_c = \frac{1}{2l} \left( \frac{T}{\rho} l_0^2 \right) n^2 \]

This can be seen from the equations for strings in paper 2, using a cross section area \( s / l_0 \), where \( l_0 \) is the belly width or bar length. Unlike a string, the soundboard is compressed, so that the frequency of an elastic soundboard is reduced, rather than increased, to

\[ f = f_c \sqrt{1 - \left( \frac{f_c}{f_0} \right)^2} \]

A measure of the effect can be expressed by the ratio

\[ \left( \frac{f_c}{f_0} \right)^2 = \left( \frac{l_0^2}{s^2} \right) \left( \frac{T}{\rho} \right) \frac{12}{\pi^2 n^2} \]

Using the previous typical values for a lute, and a tension of 3kg in each of the thirteen strings of a seven course lute, gives \( \left( \frac{f_c}{f_0} \right)^2 = 2.8/n^2 \). This seems alarming since it indicates that the effect of compression might be comparable with the bending, and could remove the first mode. Some resolution can be found from the case of a purely static bending beam (eg Ref 1). When a beam is compressed at the ends by forces that act along the axis, there is a critical load above which any small lateral displacement cannot be resisted. This is given by

\[ T = \frac{E}{s^2 l_0} \left( \frac{\pi^2}{12} \right) \]

This is also exactly the point where \( f_0 = f_c \), so no vibration is possible and any slight lateral movement would lead to collapse. The critical force has a typical value of 13.5kg, or just four or five strings. For the less realistic condition of clamped ends, the critical force would be 54kg. In practice the belly does not buckle, and a main function of a string instrument is to combine stability with lightness and a range of resonances. This requires special structures of belly, bars and body to prevent collapse. The total tensions for seven and thirteen course lutes may be 39 and 72kg, with considerable safety factors before failure.

In order to account indirectly for the increased strength of a real lute compared with a simple plate, the maximum effect of string tension on longitudinal vibrations might be estimated by using a value of 13.5kg. By definition this removes the first mode, but a safety factor would just
reduce its frequency. The calculations in the previous paper for the lute with a bar spacing of 5cm can be revised to

\[ f_l = 53, 139, 259, 412, 599, 820, \ldots \text{Hz} \]

The frequencies are lowered by about 15Hz. When these \( f_l \) are combined with the transverse \( f_t \) involving the bar vibrations, the resultant frequencies are

\[ f = 220, 226, 260, 340, 467, 638, 849, \ldots \text{Hz} \]

The previous values were

\[ f = 221, 230, 268, 350, 479, 650, 862, \ldots \text{Hz} \]

These are relatively small reductions, of about a semitone, but much larger than string inharmonicity. The real effects may be lower since string tension is only applied down the central region of a belly and not the outer parts, and may also be relieved by the rose and bars. An opposite effect, increasing frequency, may occur between the bridge and tail end, where the soundboard is under tension. The overall influence on frequency may be less than the initial uncertainty between hinged or clamped end conditions. The higher resonances for separate panels would be affected only slightly since \( \ell \) or \( \omega_n \), and hence \( (f_c/f_t) \), is much smaller.

For the lute with a 10cm bar spacing the situation is similar. With an early guitar, the longitudinal vibrations are the basic low resonances, and the result of compression by the strings could be much greater. There might be serious changes to a first mode at 50Hz and to a few higher modes. The fanned bars of a modern guitar would lessen any effects of compression. Other types of string instrument may show similar effects.

The main conclusion is that any detailed programme of tests or computer runs on separate components, or even whole instruments, without tensioned strings may be indecisive or misleading. The main work should concentrate on a fully strung lute, which is a more demanding task. However, an instrument under a specified tension could still be considered as having a characteristic set of resonances. This is a further example of theory helping to design tests. Another important reason for including this analysis is its development into a theory for the structural stability of lutes and predictions of barring, which forms part 2. Before this, it is useful to discuss the standard analyses of inharmonicity for a solid string.

**Elastic bending of strings**

This treatment will provide economically some useful background to the work on strings in papers 2 and 3. The increased frequency caused by elastic bending is given by

\[ f = f_c \sqrt{1 + \left(\frac{f_t}{f_c}\right)^2} \]

A measure of this effect is the ratio \( f_c/f_t \). For comparison with the new result above for a plate, the ratio for a string can be expressed as the inverse

\[ \left(\frac{f_c}{f_t}\right)^2 = \frac{1}{\left(1 + \left(\frac{f_t}{f_c}\right)^2\right)} \]

This is usually very large, because \( d \) and \( E \) are small, so that the elastic correction \( f_t \) to the main \( f_c \) is very small, but aurally very significant. Problems can arise with thicker bass strings since \( d \) increases while \( E \) may not be small enough, and the effect increases for shorter strings and higher harmonics \( n \). A string made of separate strands can have a smaller effect, as discussed in paper 2. (For solid strings of different diameters to have equal inharmonicities, \( E \) would need to change as \( 1/d^4 \), or \( f^4 \). This might imply an even stronger modulus criterion, but this should not be required in practice. A similar impractical factor of 256 could have been involved in completely reversing the initial stretches of treble and bass strings in paper 3.) Although this theory can describe the dependences of a small negligible effect, the precise level at which it becomes musically significant will depend on exact end conditions, such as any departure from the usual assumption of a free support or hinge. This is clearer if a string is considered as a bending rod, and a tension is applied gradually. Detailed experiments and listening would be needed to assess the actual inharmonicity and falseness. In the analogous situation for a soundboard, a large difference between a clamp and a hinge was recognized, but there was no need for a crucial accurate comparison with another process.

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2. STRUCTURAL STABILITY AND BARRING

1. Introduction
The original and main purpose of bars is stability against buckling under the tension of the strings. In view of the attention given to the important, but secondary, acoustic effects of bars, it would have been unfortunate to neglect the primary strengthening. A theoretical approach might also give useful relationships between the two effects. Bar sizes and positions may be dictated mainly by stability, and this would limit the range of acoustic possibilities.

Weakly barred lutes collapse with the bridge area rising and the middle region towards the rose sinking. This is a longitudinal buckling, but it is not immediately clear how the transverse bars can prevent it. As with the acoustic work on bars, several attempts have been necessary. Theoretical engineers may be familiar with related problems, but the unusual design of a lute may have no counterparts. The goal of high strength in one direction is combined with a large volume and surface, extreme lightness, and resonance. In contrast, most engineering situations require general strength, reasonable weight and no resonance. (Notable situations are the resonant destruction of a suspension bridge by a cross wind, the need for infantry to break step on bridges, and a similar initial problem with London’s Millennium Bridge.)

2. Buckling of a plate with a bar
The starting point was the longitudinal buckling of a beam under compression in part 1. This gave a very low critical load, but with no role for the bars. An eventual explanation came from considering a static downward force, or weight, applied to the centre of a hinged or supported beam. Here the force needed to make a deflection \( y \) is \( 4E/(s/\ell)h_y \), where symbols have the meanings in paper 4, of beam or plate thickness \( s \), length \( \ell \) between the hinged ends, width \( h_y \), and modulus \( E \). (Similar relations were used to measure moduli for pine and other woods for the acoustics in paper 4.) Next, imagine a laterally extended plate with all the edges hinged to the body, and an attached bar across the centre, which resembles a simple barred belly. The force needed for the same deflection \( y \) is now the result of three components: the longitudinal part above; a transverse component \( 4E/(s/h_y)l_y \); and a further contribution from the bar of \( 4E/(b/h_y)w_by \), where \( b \) is the bar depth, and \( w_b \) its width, as in paper 4. Therefore, when the string tension produces longitudinal buckling a new stiffness or effective modulus operates. The transverse part is negligible, and the longitudinal part was found in part 1 to be insufficient, so that the necessary stability must be provided mainly by the bar. This bar acts with an effective modulus

\[
E_f = E/(b/s)h_y/(b/h_y)w_b/(l/\ell)
\]

This can now be used in the relation for a critical string tension \( T \) to cause buckling, in part 1 above. The enhancement in effective stiffness can be very large because it increases strongly as \( b^3 \). Although the transverse modulus of the soundboard is negligible, the lateral effect of the soundboard would be very important in being firmly attached to the other parts. This allows the rigidity of the bar to be effective. A key factor is the strength of glued joints, especially between a bar and soundboard, and also the bar ends at the body sides. These need to have sufficient area of contact and, although glue is strong, there must be freedom from flaws, cracks, sharp edges and deterioration. In turn, the bar will reinforce the weak cross-grain of the soundboard. The orientation of bars and bridge across quarter-cut pine for the thin belly is also useful, since several alternatives could fail in shear.

For an exploratory example, the relations above for a critical string tension for buckling, together with typical values for \( b, w_b, l \) and \( h_y \) of 15, 4, 450 and 300mm as used in paper 4, give a critical tension of 255kg. The enhancement in effective modulus due to the bar is 19 fold. This tension is 6.6 times the 39kg of thirteen strings, which would indicate an unnecessarily large effect of just a single bar in reinforcing a lute.
3. Gradual bending of plate and bar
The actual loading on a lute is not solely a compression. Since the strings are raised slightly above the belly there is an applied bending moment $aT$, where $a$ is the action height of the strings at the bridge. The amount of longitudinal bending will increase to give a balancing elastic moment of $E_f s^2 h / 12R$. Here $R$ is a measure of the radius of curvature of the bending, which can be expressed as $l^2 / 8k$, where $k$ is the maximum depth of bending from a flat plane.

The final prediction for this degree of bending in the soundboard is

$$k = (3/2) \left( \frac{E_f}{E_t} \right) \left( a \frac{l^2}{s^3} h \right) = (3/2) \left( \frac{E_f}{E_t} \right) \left( a \frac{l^2}{b^3} h \right)$$

With thirteen strings $T$ is 39kg, and $a$ is about 5mm. For a soundboard without any bars, $E_f$ in the first expression is set equal to $E_t$ or 14GPa, and the predicted degree of bending $k$ would be 17mm. This progressive bending is very large, and far more important and sensitive than an abrupt buckling which would occur at a much higher critical compression. Next, a single bar providing a modulus enhancement of 19 reduces the distortion to a more acceptable 0.9mm. Five such bars could only allow a barely perceptible 0.2mm departure from flatness. This amount of barring is very similar to practical lute bellies. From this new quantitative analysis, the design of a whole belly can follow by applying practical sense, and some further investigation is worthwhile.

4. Practical predictions
The use of a single large central bar just below the rose would not be sufficient, because the bridge needs to be a relatively stable end point. This might be achieved by another 15mm deep bar with a similar $E_f$. The position would need to be a balance of closeness to the bridge, without a significant reduction in forcing the high resonances of separate panels. This bar would also be most useful across the widest part of the belly. In practice this places the bar just in front of the bridge, rather than the narrower side towards the tail with the minor bars. There are now two bars with predicted size and position.

The two sides of the rose hole need joining and supporting with a similar or smaller third bar. The delicate details of a rose would only need bars similar in depth to the soundboard. Some degree of distributing the strengthening over several bars can be expected to be safer than relying on very few bars. The region above the rose could be fitted uniformly with two similar bars. Here, the narrower belly width indicates that the bar depths could be reduced in proportion to provide the same effective modulus, as seen from the relation for $k$. The upper bar might be halved in depth. Bars further from the central bar might be considered to have a smaller displacement and so need a smaller stiffness, but this effect would be small because support is required towards the bridge.

There is now a five bar arrangement, very similar to the drawing by Stephen Barber referred to in paper 4. This early lute had 10cm spacings between the tail, the lowest bridge bar and the central bar. For more uniformity, an extra bar between the bridge bar and the central bar, and another in the neck region, give the more usual seven bar scheme with several 5cm spacings. The theory accounts for the way a weakly barred lute collapses, and a similar effect when a lower bar becomes detached from the soundboard or body sides. It can also be seen that above some low limit for the degree of bending $k$ all the other joints would become threatened.

5. Relation between stability and acoustics
The relation of structural barring to the lute acoustics could be discussed at length, but the following main points may be sufficient.

(i) The bar depth $h$ is even more important for stability than for the resonances. Resistance to bending, say $B$, can be seen to depend on $b^2 w h n$ for a given size of lute, where $n$ is the number of bars, or a sum over different sized bars. The mass of the bars, say $M$, varies as $b w h n$. The lowest transverse frequencies $f_t$ were found to vary as $b$, and now for an averaged effect of
several bars as $\sqrt{(B/M)}$, plus the previous factors for bar modulus $E$, length $l_b$, and added mass of soundboard. The ratio $B/M$ is also the strength to weight ratio, which in most engineering situations is preferably high. For a lute in contrast, a low value of $B/M$ is required to give low values for $f$, together with low values of $M$ for an effective forcing by the strings. This means that $B$ also needs to be as low as possible. This indicates a fundamental supporting reason behind some practical feelings that lutes should be light, and almost at a point of collapse, for a good sound. For instruments with other types of geometry, the overall strength does not depend so strongly on the elasticity of a vibrating component. Examples are violins and banjos with strings attached at the tail, rather than at the belly as for lutes and harps.

(ii) A striking practical consequence of the strong dependence of stability $B$ on bar depth $b$ is that a smaller 10mm deep bar has an enhancement factor, $E_Y/E_b$, of only 5.6. Seventeen such bars would be needed to replace five 15mm bars. The mass would be increased 2.3 times, requiring stronger driving, and the belly would begin to resemble a lumpily thickened soundboard. The low $f$ and $f$ are decreased, but any resonance of separate panels would be shifted to uselessly high frequencies. Conversely, a larger 20mm deep bar has an enhancement 8 times a 10mm bar, or 2.4 times a 15mm bar. Only two 20mm bars with about half the mass would be required, but this allows little scope for spreading the strengthening effect. This predicts that for a uniform array of relatively few bars, 15mm is a good intermediate bar depth. Much smaller depths, while not useful for general stability, are sufficient for rose details and near the neck. However, one or two main bars made slightly deeper than 15mm could ensure stability, for little extra mass. A slight rise in $f$ might be offset with a slightly smaller bar width $W_b$. Real lutes are known to have bar depths ranging between 12 and 18mm, with several smaller rose bars, and sometimes a main central bar over 20mm.

(iii) The preceding conclusions can be generalized. If the stability $B$ is kept constant while bar depth is reduced in order to reduce $f$, then increases in mass $M$ and hence $W_b$ are needed. If both $B$ and $b$ are fixed then so is $M$, and hence any increase in bar width $W_b$ will require fewer bars $n_b$. This may slightly increase $f$, since the extra effective density from the soundboard may vary as $1/(n_b + 1)w_b$, which decreases a little for wider bars. The early conjectures on squarer bars in paper 4 have only a limited region of support in this later work, since the bar depth has a dominant effect. If different sizes of bar are used, the various stiffnesses and masses might be averaged to give an effective $f$. It is possible that a uniform variation down the length of a lute would be desirable, with lighter bars near the neck, but a very large central bar might be unhelpful. A treatment of all the factors would be lengthy, but the relations could be used to assess existing and proposed designs.

(iv) Five lutes with readily available plans can be examined, and the calculations are listed in the table. The five bar drawing by Stephen Barber has the lowest values for $B$, $M$ and $f$, just below the five bar reference case. The seven bar drawing by Philip MacLeod-Coupe has $B$ and $M$ values about 30% higher. The other three lutes have a large 24mm deep main bar, which causes a large increase in $B$, with little increase in $M$. Slightly heavy bars were suggested as a starting point by Michael Prynne, but would have seemed very light forty years ago. The Harwood and Isaacs kit had a 6mm wide main bar which increases both $B$, $M$ and also $f$. These comparisons do not include the relatively small differences in lute size, but the Frei lute, drawn by Paul Thompson, is larger and closer to an F lute. The last three lutes are also pear-shaped, whereas the first two are more rounded. The relation for $k$ shows that slimmer lutes, with a smaller $b/l$, can have greater stability. This would need to be balanced against higher belly resonances, and the Helmholtz resonance, which is higher for smaller volumes, $b^2/l$. The last two columns are discussed in paper 6.
6. Further variations, lute sizes, and other instruments

(i) The prediction of a high sensitivity of stability to bar depth has some remarkable consequences. The earlier choice of a 15mm bar depth to illustrate the acoustic work has been very fortunate, but this later stability work has not been massaged into agreement. Although there would be a progressive increase in the degree of bending \( k \) with decreasing bar depth, rather than a critical condition, the variation with \( b^3 \) is so strong that any reasonably small value of \( k \) predicts practical values for \( b \). For example, halving the depth of 15mm for five bars gives a much larger 1.6mm degree of bending, while doubling the bar depth gives an unnecessarily small 0.025mm. For smaller variations, a 10% change in \( b \) produces a 30% opposite change in \( k \). If a doubling of bending from 0.2 to 0.4mm were acceptable, the allowed reduction in \( b \) would be 20%, or five bars 12mm deep. This suggests that any substantial paring down of the light barring in the Stephen Barber drawing might be risky.

(ii) Another important influence on the degree of bending \( k \) is the action height, \( a \), of the strings at the bridge. This needs to be low to reduce the applied bending moment. Greater values would also cause a tipping of the bridge towards the nut, with unacceptable local distortions of the soundboard. The width of a bridge, with a large area of contact on the soundboard, and a sloping back are also helpful.

(iii) The way in which stability will scale with the size of lute, or \( l \), can be examined from the relative degree of bending (\( k/l \)). If all the dimensions, \( b \), \( a \), \( b \), \( w_b \), in the equation are proportional to \( l \), then for equal values of (\( k/l \)) the string tension needs to change as \( l^2 \). This appears to be an important result. Larger lutes with no increase in tension are predicted to have the advantage of greater stability. Alternatively, they could have considerably greater tensions, or action heights, or have bars of relatively smaller depth and width or greater length, or any desired and allowed combination of the different types of change. Conversely, smaller lutes need to obtain strength by increasing \( b \) or \( w_b \), or by reducing \( T \), \( l_b \) or \( a \). These mechanisms, therefore, also show that larger lutes can have relatively lower resonances, particularly the low transverse \( f \), by reducing \( b \) or \( w_b \), and increasing \( l_b \). This shows a fundamental reason why larger lutes can be made relatively lighter, fatter, and more sonorous, without losing strength. A greater sonority for larger lutes was also possible on purely acoustic grounds in paper 4. These strong effects of size on optimizing and designing lutes are examined further in paper 7. It is worth noting the contrast with many engineering situations where loads increase as weight or \( l^3 \).

(iv) The ideas can be applied to other string instruments. The strength of a light early guitar or vihuela would depend on the two centrally placed bars, and the highly tensioned modern guitar would have a direct longitudinal resistance from the fanned bars. The shallow sides, flat back and narrow waist might provide greater strength than the deep curved body of a lute. Violins and the banjo also have these features, and the strings are fastened to the tail, rather than a flexible belly. At another extreme, the strings of a harp are tensioned perpendicular to the soundboard, which therefore needs to be narrow, and perhaps thicker, for stability. The relations found above between \( B/M \) and \( f \) for the lute, would not hold for a banjo; probably not for a violin; maybe in a weaker form for a modern guitar; in a slightly stronger form for an early guitar; and in an extreme form for a harp. Predictions of resonances for harps can be made from paper 4. The main differences from the lute arise from a longer, narrower, thicker soundboard, a wide variation of string positions, and a widening soundboard for the lower strings.

This analysis appears to give a satisfying and relatively simple result for the important problem of lute stability. The quantitative treatment of barring schemes is almost like studying an evolved natural system. The original makers would have developed their designs by continuous exploration and seen the sensitivities, but without the \( (b/l_b)^3 \) formula. Experiments could test the theory further by measuring degrees of bending for various tensions and barring schemes. This static problem might be a suitable start for any computing on lutes.
Stability and Resonances

<table>
<thead>
<tr>
<th>Lute drawing</th>
<th>Number of bars, ( n_b )</th>
<th>Stability factor, ( B )</th>
<th>Mass of bars, ( M )</th>
<th>( \sqrt{(B/M)} ) related to ( f_t )</th>
<th>Bridge bar depth, ( f_b )</th>
<th>Lower panel, ( f_p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reference each bar 15 x 4 mm</td>
<td>5</td>
<td>67,500</td>
<td>300</td>
<td>15.0</td>
<td>15.0</td>
<td>7.9</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>8.8 (5)</td>
</tr>
<tr>
<td>Barber (Lute Soc drg)</td>
<td>5</td>
<td>58,000</td>
<td>280</td>
<td>14.3</td>
<td>15.0</td>
<td>7.6</td>
</tr>
<tr>
<td>McLeodCoupe (Lute Soc drg)</td>
<td>7</td>
<td>79,000</td>
<td>350</td>
<td>15.0</td>
<td>14.0</td>
<td>8.2</td>
</tr>
<tr>
<td>Frei (Warwick Museum drg)</td>
<td>6</td>
<td>95,000</td>
<td>345</td>
<td>16.6</td>
<td>12.0</td>
<td>6.0</td>
</tr>
<tr>
<td>Prynne (Lute Soc J. 1964)</td>
<td>6</td>
<td>106,000</td>
<td>350</td>
<td>17.3</td>
<td>12.7</td>
<td>(6.3)</td>
</tr>
<tr>
<td>Harwood and Isaacs</td>
<td>7</td>
<td>144,000</td>
<td>405</td>
<td>18.8</td>
<td>12.7</td>
<td>6.7</td>
</tr>
</tbody>
</table>

3. THE CONCAVE SURFACE OF RIBS

A striking feature of many lutes is the concave or hollowed surface of the individual ribs. There appear to be several explanations in circulation, including (i) a hot iron to bend the ribs; (ii) steam for bending; (iii) soaking for bending; (iv) the angled edge on ribs; (v) gluing the edges together; (vi) scraping out the hollows; and (vii) a mistake in construction. Most people find the appearance attractive, and the angle between the edges highlights the contrast of concave segments on a convex body. Technically minded people would think that elastic and other mechanical properties of wood are responsible. The treatment of bending in the stability work suggested an analysis of the rib hollowing.

If an elastic plate or strip is held by one pair of edges, and then bent, an opposite curvature is produced across the free edges. This depends on the Poisson’s ratio \( \sigma \), which is a measure of the strain or length change in one direction when stress is applied in perpendicular directions. This ancticlastic (opposite breaking) curvature is a century old method for measuring \( \sigma \), as an alternative to the contraction of a stretched sample (eg Ref 2). If the radius of curvature of the main bending is \( R_b \), then the radius for the ancticlastic hollowing is given by \( R_a = - R_b/\sigma \). The forces in perpendicular directions are sums depending on strains in both directions, so for a zero force or moment across the strip there must be opposite strains or curvatures, and hence the negative sign for this saddle shape. Values of \( \sigma \) must lie between 0 and \( \frac{1}{2} \), and are typically 0.25 to 0.4. For isotropic materials the effect is scarcely noticeable and tests require optical methods.
The wood used for ribs is far from isotropic, so the analysis needed extending to give

\[ R_a = - \left( \frac{R_b}{\sigma_t} \right) \left( \frac{E_t}{E_i} \right) \]

Here \( E_t \) and \( E_i \) are the elastic moduli along and across the rib, similar to the soundboard, and \( \sigma_t \) represents the contraction in width when the rib is stretched lengthways. The radius \( R_b \) can be expressed roughly in terms of the length of a deformed arc, which is just the body length \( l \), and also the depth \( c \) of the arc or body, as \( R_b^2 = \frac{l^2}{8c} \). This is similar to the previous stability analysis, and for the depth of hollow \( h \) on a rib of width \( r \), \( R_a^2 = \frac{r^2}{8h} \). The predicted degree of rib hollowing is therefore

\[ h = c \sigma_t \left( \frac{E_t}{E_i} \right) \left( \frac{r}{l} \right)^2 \]

Using typical values of 45cm for \( l \), 4cm for a wide rib on a nine-ribbed back, 12cm for \( c \), 0.3 for \( \sigma_t \) and 3.8 for \( (E_t/E_i) \), gives a hollow \( h \) of 1.1mm. The depth of hollowing seen on many lutes with relatively few ribs is close to 1mm. The calculation depends strongly on the value of \( (E_t/E_i) \), which I measured on a square of sycamore. The ratio of weights required to cause the same deflection for the two orientations of supporting opposite edges gave a direct value. This avoided the general treatment of loaded beams, which had been needed for the soundboard tests. The value of 3.8 is far less than 28 for quarter-cut pine, which would have a predicted hollowing seven times greater, implying fracture and unsuitability for ribs. The prediction is not very sensitive to \( \sigma_t \). Both \( R_b \) and \( R_a \) are close to 20cm, which is not obvious for \( R_s \). For harder woods, both \( \sigma_t \) and \( (E_t/E_i) \) tend to be smaller. These woods are generally used for narrow ribs, so that any hollowing will be smaller and also less prominent. Harder woods are also seen for thicker ribs, as on some modern ouds, with the joining edges smoothed off to give a completely convex body.

This analysis appears to give a good quantitative explanation of concave ribs. However, it is the heating, by any method, that changes the wood structure into a plastic state, where the fibres can slide and allow bending rather than fracture, and then cool to a set shape. Similar relations, accounting for incompressibility, may also hold here, and the process of bending could also be analysed as a series of small reversible steps. It would be interesting to test wood in a normal state, then during heating and cooling, the joining of the edges, and any effect of the tapered outline of a rib.

References
FURTHER ACOUSTICS OF LUTES:
BRIDGE AREA, MINOR BARS, ANGLES (6)

Before the main work involving the barring I had thought that the key sensitive region of a lute was around the bridge, where the strings drive the soundboard. Several analyses were made, but with the more positive treatment of bars, the bridge was given just a few remarks in paper 4. This communication is a renewed focus on the lower belly. The two aspects can be reconciled in a practical way, which also reduces any large distinctions between lutes with different bar spacings. Later sections deal with the small treble and bass bars, and then the effects of angling the main bars and a bridge. Theory is pushed to the limit in trying to understand these tantalizing details. Although it is difficult to be firm, and the results are not expected to be accurate, similar lines of inquiry could also form a plan for experiments.

1. A balanced system
One might expect the response of a lute to depend on the following factors, in an equal or decreasing importance: (i) the player’s fingers, (ii) the strings, (iii) the bridge, (iv) the belly around the bridge, including the bridge bar and minor bars, (v) the air inside the body, (vi) the rest of the belly, and (vii) the body and neck. For example, if (ii) were poor then good later factors could not provide a useful correction. This differs in emphasis from some views that a good lute shines through poor strings and a mediocre lute is scarcely improved with good strings, or that bridge areas are not too important. An analogous situation arose just before the demise of vinyl records, when it was realized that there was a decreasing importance of stylus, cartridge, turntable, amplifier and speakers. Within any quality of balanced system this shift of attention from amplifiers and speakers could be demonstrated vividly by replacing a superb stylus with a moderate one. In effect, no expense on later components would remedy a poor input. This may be worthwhile thinking for the series of related processes in a musical instrument. If there were no validity, then the transforming effects of the later stages would be more significant than the quality of input.

2. Active lower panel
Most types of lute have their bridge placed about \(1/6\) of a body length from the tail end, which is about 7cm for the typical 45cm long body of a G lute. The bridge bar, just above the bridge, is about \(2/9\) or 10cm from the tail. This highly active area of length 10cm, with a large width of about 30cm, is essentially the same for lutes with different spacings of the upper bars. On this view, all the low frequency overall modes of the transverse \(f_t\) and longitudinal \(f_l\) predicted in paper 4, would be characteristic of the common 10cm spacing, but with a little bias to the slightly higher values for 5cm spacings in the seven bar design. The resonances of separate panels would also be mainly the 10cm values for the lower panel, even for lutes with closer upper bars. The slightly heavier, stronger lute with 5cm spacings would have some extra, clearly defined higher resonances. The lighter lute with 10cm spacings would be easier to force, and faster in response. At a general resonance, the amplitude builds at a rate \(F/Mf\), which is greater for a larger force \(F\), a smaller combined mass \(M\) of bridge, soundboard and bars, and for lower frequencies \(f\).

For analysing this lower area around the bridge it is highly instructive to estimate the length of a free panel of soundboard that would be needed for a longitudinal \(f_l\) to become equal to the low transverse \(f_t\). Both these frequencies depend strongly on the length of a panel, and the length of a bar. This critical length \(l_c\) of soundboard free of any bars is estimated to be about 16cm, by
considering the largest previous $f_i$ and the lowest $f_i$ in paper 4 (or $45\sqrt[22/174]{(22/174)}$). A general relation can be deduced from the equations for $f_i$ and $f_t$ as

$$\frac{l_c}{l_b} = \frac{(s/b)(l_b/w_b)^{1/3}}{b}$$

which gives 17.6cm. There is no dependence on material properties because both $f_i$ and $f_t$ depend on $E$. This confirms that vibrations of the bridge bar at 10cm from the tail, and hence within this critical length, are vital for determining the lowest resonances. An additional bar about 5cm higher may be less active, but a large central bar about 20cm from the tail would not have a crucial effect on the low resonances. This may remove an earlier doubt about such enormous bars, in paper 5 on stability. This local influence of the lower panel also reverses the earlier idea of the low resonances being determined by an averaged effect of all the bars.

For lengths above $l_c$, the longitudinal $f_i$ decreases rapidly and so cannot greatly increase the resultant resonances produced from combinations with the low transverse $f_t$. There may be a reason here behind practical rules for positioning a rose hole. Ref 1 places a rose centre at a distance from the tail that is 0.85 times the body width, and Ref 2 gave a 1.0 factor for longer pear-shaped bodies. This is not easily understandable from the function of the Helmholtz resonance. However, one effect is to keep intact an important large lower area of soundboard. A further effect is to prevent a large central bar just below the rose being too close to the tail, which would greatly raise the low resonances, by a factor of $\sqrt{2}$ as $f_i$ approaches $f_t$. In general, bunching up the bridge and bars towards the tail end would produce higher resonances, and spreading them upwards will give lower frequencies, but not below the $f_t$ determined by the bars. This is roughly analogous to the position of plucking a simple string.

The effects of the lower panels can be illustrated for the six lute designs in the table of paper 5. These initially provided examples of barring for stability, and the clearest comparison is a wide range of stability but smaller variations in the measures of $f_t$. The acoustics should not be taken as a ranking of quality, and it could appear even more invidious to treat six fine old lutes. The sixth column simply takes the low transverse $f_t$ as proportional to the depth of the bridge bar. The last column includes the effect of lute width and the added mass of the panels. The reference case shows the effect of a 5cm panel above the bridge bar, which also occurs in the last four designs. This increases $f_t$, but the larger Frei lute has a lower panel 13cm long, which acts to reduce $f_t$. A short 5cm second panel in the seven bar designs may be able to provide stability with a smaller depth of bridge bar, which would be an advantage for both stability and acoustics. This changes the measure of $f_t$ in the Harwood and Isaacs with a large central bar, but not in the MacLeod-Coupe lute. Unexpected ideas may be the best result of such comparisons.

In summary, the lower half of the soundboard around the bridge, and the lower bars, are responsible for the low overall resonances characteristic of a large bar spacing of typically 10cm; and also for the higher resonances of panels with the same spacing. There is a smaller added effect from any closer spacings higher up the belly. These effects for the lower active panel, common to all types of lute would fill the gaps of moderate frequencies anticipated in paper 4 for lutes with 5cm bar spacings. This also suggests only moderate differences between different schemes for main bars. The particularly sensitive area would be the bridge, the bridge bar, the intervening strip of soundboard, and the other minor bars.

### 3. Bridge position

In paper 4 it was suggested that the position of a bridge would not greatly affect the forcing of low overall modes, so that a bridge close up to the bridge bar could be as effective as in a lower position. However, longitudinal modes that would need a node near the antinodal bridge can be expected to be weak. A bridge placed at a nominal 10cm from the tail indicated weaker fourth and fifth longitudinal modes based on the 45cm body length. Now at a detailed 7cm, the sixth mode at 705Hz for the 10cm bar spacing, and also the seventh mode at 947Hz, would be weak.
These simple geometric details can be applied to the added mass of the main bars in paper 4. For the 5cm bar spacing, the third and sixth modes will have a quarter of the bar mass at the nodes, and hence ineffective in reducing \( f_1 \). This would slightly increase the resonances at 268 and 650Hz to 273 and 684Hz. For the ninth mode, all the bar mass would be at the nodes and the resonance at 1394Hz would be increased greatly to 1796Hz. These effects may occur with perfect repeats, but are less likely with mixed repeats, which is implied by a 10cm spacing on a 45cm length.

In contrast, the longitudinal panel resonances could be very sensitive to bridge position. If the bridge were 5cm from the tail, in the middle of a 10cm panel, then odd harmonics would be strong but even harmonics absent. A bridge close to the bridge bar could produce many harmonics, but only weakly, analogous to a string. The typical bridge position at about a third, or 3cm, from the bridge bar appears to be a good compromise, giving moderate first and second harmonics, and weakening only the third and its multiples.

For both types of transverse resonance, the effect of a bridge could be very complex, depending on its detailed motion, relative stiffness, and the position of the forcing string. For example, a first mode may be stronger for centrally positioned strings such as a fourth course, and a second mode would involve a rocking motion driven by side strings such as trebles and basses. Higher modes could be weaker and depend even more on string position. The combined motion of soundboard, bridge and the minor bars is examined in the rest of this paper.

4. Bridge mass

If the bridge is analysed as a mass \( m_b \), concentrated near the centre of a soundboard or panel of mass \( m_s \), the frequencies are reduced by a factor of about \( \frac{1}{\sqrt{1 + 2m_b/m_s}} \), as indicated in paper 4. For example, a typical bridge 12cm long, 8mm high, and 12mm wide, on the previous typical soundboard, would slightly reduce both the overall \( f_1 \) and \( f_2 \) values by a factor of 0.96.

However, if the considerable length of a bridge is treated similarly to the added mass of a main bar, the factor of 2 is effectively removed in the above relation. For a bar spacing \( w_s \) of 10cm, the overall \( f_1 \) and \( f_2 \) would be reduced by a factor of 0.93. This doubled effect also results from \( w_s \) being about one quarter of the body length. If the lower active area is taken as about two panels, the two treatments lead to a common factor of 0.96. These estimates can be seen as two limiting cases, of a point or a line of mass, for the effect of a bridge. In the work below on separate panels, a bridge will be treated by the first method, and minor bars will use the second.

5. Methods of analysis

The various types of analysis may interest the technically minded. Before the main work on bars, I made several theoretical attempts with a bridge on a soundboard, or an added mass on a plate, with a string or force on the mass. These ranged from complete exact solutions, through a variety of approximations, to a method due to Lord Rayleigh (Ref 3). In this method the vibrating system is treated as statically distorted, and a simple oscillation is assumed. Then the kinetic energy and the potential energy of bending are found by integration, and equated to give the frequency. For a simple beam, this method is remarkably close to the exact solutions.

The soundboard work in paper 4 started with exact solutions and then added approximations. The stability analysis in paper 5 was in effect a simple static solution, which also had a relation to the acoustics. These simplifications are used in the next sections to examine the acoustics of a soundboard with a bridge and minor bars. The Rayleigh method involves much calculation, but is still unable to predict higher modes, which depend on end conditions. Exact boundary conditions are unknown for a lute, but the approximate methods could still be used to explore the effect of some selected intermediate cases between a hinge and a clamp. Clearly, since there are so many complications, the simplest approach is necessary. Also, a single exact topic would not match well with more intricate, vague or approximate aspects.
6. Treble bars

Further details near the bridge are the small treble and bass bars, and the very brief remarks in paper 4 need expansion. A final bonus in section 13 has been an analysis of the curious feature of angled bridges and main bars on some old lutes.

It is necessary to be as specific and practical as possible, and at least five aspects are useful. These are: (i) direct effects on soundboard resonances; (ii) details of the bridge shape and motion for different string positions; (iii) a convenience for making fine adjustments; (iv) possible structural effects; and (v) historical developments involving extra strings. It is best to begin with the earlier lutes, rather than struggle with the later more familiar complicated designs. A single treble bar angled at about 45° extended from under the treble string on the bridge to the lower side of the body, as in Ref 1. Perhaps even earlier lutes had no treble bar.

Firstly, the effect on the low overall modes, controlled by the large main bars, can be only slight. A small bar about 3mm wide and 4mm deep, unangled and lying purely across the soundboard grain, would very slightly increase the low transverse frequency $f_t$ and reduce the longitudinal $f_l$. Such a bar lying purely along the grain would have the opposite effect, so that for an angled bar the slight effects would also tend to cancel out.

Near the bridge the treble bar is similar to the scalloped end of the bridge, and both these details could reduce the amplitude of motion possible near the treble strings. The widths of flexible soundboard at each side of the relatively rigid bridge will lead to less resistance to up and down vibrations at the ends of the bridge than at the central region. The difference could be less for a back and forth tilting of the bridge, which would be produced by sideways plucking, as discussed in paper 4. Next, the higher resonances of the lower panel can be shown to be more greatly affected by the treble bar and bridge.

7. Bridge with an unangled treble bar

Firstly, the effect of a bridge alone must be estimated, and its mass will reduce frequencies. In contrast, a rigid bridge of length $L$ will reduce the free width of soundboard at the sides and so increase its stiffness and frequency, as suggested in paper 4. The basic equation for the longitudinal $f_l$, based on $E_t$ and $w_s$, can be reduced by the factor $1/\sqrt{1 + 2m_b/m_s}$, where $m_b$ is the bridge mass used above, and $m_s$ is $(\rho_s w_s L)$ for a panel. For the transverse $f_t$, based on $E_t$ and $l_b$, a similar reduction is made, but with a central mass of soundboard ($\rho_s w_s L$) added to $m_b$.

In addition, the effective width of vibrating soundboard is reduced from $l_b$ to $(l_b - L)$.

For the bridge dimensions in section 4, on a panel 10cm long, the longitudinal $f_l$ is reduced by a factor of 0.84 from 445 to 376Hz. This is about four times the effect of the bridge on the low overall resonances. For the transverse modes, the same reduction is exceeded by stiffening, and $f_t$ is increased by a factor of 1.54 from 9.2 to 14.2Hz. The resultant resonances, from combining longitudinal with transverse modes (x 4, 9, etc), are

$$f_{ba} = 376, 380, 397, 439, 517, 634, 791, \ldots \text{Hz}$$

These can be compared with the values for no bridge in paper 4:

$$f = 445, 447, 453, 469, 501, 555, 633, \ldots \text{Hz}$$

As before, these lists are a good way to illustrate predicted trends, but no certainty is implied by the precise numbers. The bridge mass reduces the basic $f_b$, but the increased $f_t$ values have the effect of spacing out several distinct resonances. The first five modes now cover a range of 140Hz, compared with a narrow bunch of 50 Hz without a bridge. Both these effects of a bridge may be desirable for producing a spectrum of low and moderate resonances. Now, the effect of a treble bar on resonances of the lower panel can be found by adapting the equations in paper 4.

For the longitudinal $f_l$: in the case of a purely transverse bar, the equation for $f_l$ with a bar has the bridge effect of $2m_b$ added to $(\rho_b w_b l_b)$ for the small bar. For a purely longitudinal bar, $f_l$ is also increased by a further factor of

$$\sqrt{1 + b^2 w_b / s^2 l_b}$$
This increase of frequency is caused by the extra stiffness from the treble bar. For the transverse $f_t$, the equation for bar vibration is recast to describe soundboard vibration. Then for a purely longitudinal small bar, the effect of a bridge is treated just like $f_l$ above. For a purely transverse bar the $f_t$ are further increased by a potentially large factor of

$$\sqrt{1 + \frac{E_b b^2 w_b}{E_s s^4 w_s}}$$

As an example, for a purely longitudinal treble bar 3mm wide and 4mm deep, $f_l$ is increased by a factor of 1.08, and $f_t$ is reduced by 0.97, showing relatively small effects of added mass and the stiffening. When this treble bar is purely transverse, the $f_l$ is reduced by the factor of 0.97, and $f_t$ is increased greatly, by a factor of 2.7. This shows the large effect of a transverse bar on the flexible cross-grain of a soundboard. This is similar to, but not so great as, the factor of 24 for a typical main bar, which was deduced in paper 4. The small effect of the mass of a minor bar means that a single separate factor can be used, and the four combined effects of stiffness of soundboard and bar, and mass of bridge and bar can be factorized. This is preferable to using a large but still very approximate equation. The higher resonances of the lower panel will clearly be much more affected by the treble bar and bridge than the low overall modes. This is because the minor bars are much smaller in relation to a main bar and the entire soundboard, than to a single thin panel.

8. Angled treble bar

For a real treble bar angled at 45°, the effects of two perpendicular bars might be combined equally. This increases $f_l$ by a factor of 1.05 from 376 to 397Hz, and increases $f_t$ by a factor of 2.65 from 14.2 to 38Hz. Resultant resonances, characteristic of the treble end of the bridge, are $f_{ba} = 399, 425, 524, 726, 1030, 1424, 1904, ...$ Hz

The longitudinal stiffening by the treble bar partly reverses the lowering by bridge mass. The large transverse stiffening gives even wider spacing for the lower modes, which may be practically useful. The bass end of the bridge had no bars in this early lute, and so the lower resonances in section 7 were labelled as $f_{ba}$. In addition to these two sets of modified resonances for the lower panel, the panel above the bridge bar could still have the initial levels. For an illustration, the $f_{ba}$ frequencies are added to the table in paper 4. An overall factor of 0.8 is also applied, to represent closely an original very light lute with a soundboard thickness of 1.6mm, and also 12mm deep main bars. It is seen how all the combined effects of a general 10cm bar spacing, the bridge mass and length, and a treble bar, are able to fill the region of fundamental string frequencies from c at 131Hz, to a top g" at 784Hz, for this example of a G lute.

An angled bar is a neat way of spreading over an area of panel the two types of effect calculated for individual perpendicular bars. A diagonal bar 3mm wide is probably represented better by a pair of 2mm bars. More importantly, the sensitivity to bar depth $b$ means that a 5mm bar is nearly twice as stiff as the depth of 4mm. Bars 2mm wide and 5mm deep would further increase the above value of $f_l$ to 418Hz and $f_t$ to 46Hz.

The details are too intricate to warrant any further analysis, and the results will now be applied to some historical schemes for minor bars. One complication is that the treble bar does not cover the whole 10cm length of panel, leaving an upper strip of free soundboard, which could be a sensitive controlling factor. However, the total effect of this small bar on $f_l$ was an increase of only 20Hz, and the large increases predicted for $f_t$ will be less sensitive to this orientation of strip than they would be to a vertical gap. Also, if a treble bar did not reach the bridge or the side of the body, its stiffening would be less.

Another problem arises from the variations of effective stiffness and mass over a panel, due to geometry, modulus and density, for bars and soundboard. The idea of parallel variations in a characteristic frequency, such as $f_{ba}$ and $f_{ba}$ above, might represent real modes with local values of $f$, or just an averaged effect. Separate local values of an increased $f_l$ across a panel may be more likely when combined with higher transverse modes. Elastic bending waves have a
velocity that depends on frequency as $\sqrt{f}$, rather than a constant as for strings and many other situations. In the stationary wave solutions for the vibrating soundboard it has been most practical to express resonant frequencies $f$ in terms of lengths $l$, varying as $1/l^2$. For the ideal condition of hinged ends there has also been a clearly defined wavelength $\lambda = 2l/n$, but this does not occur for clamped or free ends.

Some slight structural strengthening by a treble bar might be useful, especially under the tip of the right hand little finger. This easily replaceable small bar could also allow fine adjustments before gluing a belly to the body, thus avoiding irreversible errors and exterior shaping.

This early scheme had no minor bass bar. There were six courses of gut strings, including only two real bass strings. These may have had a restricted dull tone, so that any moderation of bass amplitude may have been unnecessary. The bridge bar could also be angled slightly to increase the length $w_s$ of lower panel on the bass side. In Ref 1 this angle is about $2^\circ$ and the extra length is 10mm, or 10% of $w_s$. This would further reduce the $f_t$ for the bass side by about 20%, from 376 to 300Hz. The resultant resonances from combination with the previous $f_t$ of 14.2Hz are

$$f_{ba,an} = 300, 305, 326, 376, 465, 593, 758, \ldots \text{Hz}$$

This is a substantial reduction in $f_t$ for the panel resonance towards the bass side. It is comparable to the initial effect of putting a bridge on a free panel, as in section 7, but without an associated increase in the transverse $f_t$. Also, the slight angle of a main bar would scarcely affect the low overall modes. This angling could give a sensitive individual adjustment to the relative characteristic values of $f_t$ values on the treble and bass sides. In addition, the bridge itself could be angled either way. There are many possible combinations of angles, with different variations of panel length and the width of free strip across the belly. The next two sections on minor bars will keep to unangled bridge and main bars. In section 13 a new effect of angle is described, which may act in combination with the present effects.

9. Bass bar with two treble bars

A later and more familiar scheme for minor bars used a pair of angled treble bars. The upper bar was more transverse, and the lower bar was angled down the belly, maybe not reaching the bridge. This would allow independent changes in the transverse and longitudinal effects, and also in the stiffness and mass. Two bars could also influence a greater area of soundboard.

There was also a long bass bar, purely transverse and midway between the bridge and tail end. It extended from near the lower treble bar and curved up to meet the bass side of the body. This J-shaped bar was also used for lutes with a few more bass strings, and even for later baroque conversions with about six extra basses. The bass bar does not meet the bridge or restrain its motion. Its purely transverse position would increase the low overall $f_t$ very slightly, and also reduce the low $f_t$. It could temper any excessive localized motion of the large lower panel, and its position on the centre line of the free area would give the greatest effect. The longitudinal $f_t$ for the panel could be reduced slightly more from 376 to 365Hz by the small mass, and the transverse $f_t$ could be increased on the bass side to about 38 Hz, as for the treble bar. This gives resultant resonances on the bass side, and also in the central region, as

$$f_{ba} = 369, 395, 500, 709, 1018, \ldots \text{Hz}$$

The lowest mode is below the previous value without a bass bar, in section 7, and the higher frequencies characteristic of mid way up the panel rise towards the treble value. However, using two treble bars could increase the effect on the treble side.

In this scheme the bridge and bass bar divide the lower panel into thirds. The bass bar is at the other node of a missing third harmonic, as in section 3 above. This means that all other longitudinal modes for the panel will be affected less by the bass bar. The first harmonic at the nominal 445Hz, and the much higher second at 1780Hz, are therefore reasonably strong. This detail, and the higher $f_t$, may help dull lower basses, which continued to require octave strings.
The bass bar would not provide much strength, apart from reinforcement across the soundboard grain.

All types of bar could be tapered in depth, and if this occurs towards the centre of the belly, the frequency may be greater than for a bar of the same average depth. Also, some bridges had an increased cross section towards the bass end, where the values for $f_l$ and $f_t$ would be made slightly lower.

10. Four small bars, and slim bridges

For later baroque lutes, the slender bass bar was replaced with two small bars. These were symmetrical with the treble bars. Together with the much longer bridge, for carrying twice the earlier number of strings, there would be considerable extra stiffness in the lower panel of a soundboard. For a bridge with a greater length of 18cm, the equations predict a reduction in the longitudinal $f_l$ for a panel from 376 to 352Hz. The transverse $f_t$ for a panel is increased to 24.5Hz, from 14.2Hz for the 12cm bridge. The increased stiffness has a stronger effect than the greater mass. The resultant resonances are

$$f_{na} = 353, 365, 415, 527, 706, 950, 1250, \ldots \text{Hz}$$

This shows a considerable lowering followed by a greater spacing of the first five modes over 350Hz.

These small bars at both ends of the bridge would have effects similar to the previous treble end, increasing $f_l$ to 370Hz and $f_t$ to a very high 65Hz and giving resultants of

$$f_{na, tr} = 371, 452, 692, 1104, 1666, \ldots \text{Hz}$$

The first two modes are quite low, but for these higher values of $f_l$, the resultants soon increase very rapidly as $m^2$, above the second longitudinal mode at 1480Hz (or 4x 370).

With this greater effective stiffness of soundboard, the bridge itself may begin to bend and resemble a main bar, as suggested in paper 4. Later bridges were also slimmer, as well as longer, which would decrease effects of mass, but increase the bending. The limiting case of a vibrating bridge can be found from the equation giving an overall $f_t$ for the vibrations of a main bar. For a bridge depth, width and length of 8, 12, 300mm, and an effective panel length of 10cm, the frequency of transverse vibrations for the bridge would be 109Hz. For a harder wood, $E/\rho$ may be twice the value for pine, giving a value of 154Hz. A 5cm panel gives values of 134 and 189Hz.

The high estimate of 65Hz found above for the $f_t$ of a panel with a baroque bridge is still far less than any limiting case. This may indicate that significant bending of bridges is unlikely.

A related effect occurs with the main bars of some baroque lutes. These bars have a depth that increases towards the centre. This will increase the stability over the centre, but the central extra mass will tend to reduce the low transverse $f_t$, since the stiffness is controlled mainly by the hinged ends. This is the reverse of tapering a minor bar towards the centre, in section 9. Any value of a complicated general analysis would be reduced by uncertainty about the end conditions.

11. Resonances for bass strings

From this analysis, it appears that baroque lutes with added bass strings also had minor bars around the bridge that would raise resonant frequencies of the panel near these bass strings. Furthermore, some bridges were offset to be closer to the bass side (eg Ref 4), which would produce higher frequencies for the bass side than for the treble side. The first few modes could still be low and well spaced, but there seems to be a reversal of the situation for earlier lutes. A possible explanation might involve two other unexpected aspects: the octave strings, and the low Helmholtz resonance at $f_h$.

With reference to a G lute, a fifth course c at 131Hz is near the strong $f_h$, which cannot have any associated higher harmonics. There may be an octave string, and also higher harmonics of the bass c, able to provide lightness by the overall modes and panel resonances. For a lower string,
such as a bass G, there is still an influence of the $f_h$ region, and maybe also lower resonances of the body and neck, as suggested in paper 4. However, an octave string to this G would also now be approaching $f_h$. For one extra bass course at D, the octave string itself is at $f_h$, and can provide support by the Helmholtz resonance. Brightness can come from only the higher harmonics of both strings in this course, acting on the overall modes and panel resonances. For a baroque lute with five to seven extra basses and octaves, there would be a gradual relative shift of the Helmholtz region from a higher bass string down to a lower octave string. For these effects it would not be vital for panel resonances to be low near the bass side. Rather, definition and brightness would be helped if these modes were quite high, but still well spaced, as predicted above by the geometry. The shorthand statement of octave strings improving dull basses would involve a combination of several effects in the lute and the strings. These details suggest that improving the low and middle resonances for several extra bass strings on a baroque lute with a long J bar became less useful or desirable. Instead, brightness was enhanced, and the shorter bars may have helped the stability under an increased tension. The minor bars at each end of the bridge could restrain the motion to a level similar to the centre, but still retain the tipping motion. These later lutes with more strings therefore appear to have a further ingenuity, similar to the angled necks and bridges with tapered strings, in paper 3. A major loss, however, may have been a rich variety of low and middle range resonances, as predicted for the earlier lutes. Larger sized bodies might have compensated for this.

There were many later schemes for the minor bars. A lute by Berr, described in Lute News 47, had just one almost transverse small bar at each end of the bridge. The bass bar was the long early type, but ran between opposite sides of the body, and was placed much lower and closer to the tail. The present analysis shows how this ‘cradle’ around the bridge would give purely transverse increases of stiffness, frequency and strength, with small reductions of longitudinal frequency from the added mass.

12. Fine adjustments

This long treatment of the bridge area has been encouraged, but not unduly influenced, by a very interesting practical account of making fine adjustments (Ref 4). An iterative removal of wood from two treble bars and a long bass bar was used to ‘tune the belly by driving a tight spot off the bridge ends’. This tight spot, with a restricted sonority, was found when the bridge was tapped at the centre. A little wood could also be taken off the bridge bar and the soundboard just below it. This personal method seems consistent with the possible motion of the bridge discussed above. This motion can be made more uniform by starting with oversized bars, and then reducing them to match the resistance at the bridge centre. There was no stated intention to adjust frequencies of resonances, which were the main interest above. However, from the experience of making hundreds of lutes, it was stated that 1mm difference in the space between a bridge bar and the bridge can produce more effect than doubling the depth of a large central main bar. This apparently astonishing remark is entirely in line with the present analyses, but I would not have been so provocative with the predictions. From brief discussions, other lute makers have similar procedures, often taught or passed on and difficult to describe, but these are not necessarily the old makers’ methods and intentions. Ref 4 also gives some practical trends for thinning or shortening a length of wood to change its pitch. Detailed quantitative versions are the basis of the physics and maths for the whole lute, and could be extracted from the present work, or listed systematically. Many of the other relations for the acoustics, stability and optimization could be tested or used as a guide by makers.
13. Angled main bars and bridges
This topic involves the intriguing observation that some old lutes had angled bridges, and even angled main bars. The final research on strings, in paper 3, gave explanations for how old strings on lutes with angled bridges and necks could be tuned. The other aspect of the puzzle is what acoustic effect may have been intended for the belly, or simply resulted from accommodating the old strings. It was noted in paper 4 that the effective elastic moduli, across and along the soundboard, would be altered by the angles. At that stage of the work, attention was on the 5cm bar spacing, with resonances of panels at high frequencies of about 1800Hz, so that the effects seemed of little interest. Two mechanisms have now been found that predict significant and useful effects, even for the small angles of about 3 to 6° seen on the old lutes. The result depends on the anisotropy of wood and the direction of the grain. It is additional to the variation in panel length across the width of a lute, which affected the longitudinal frequency \( f_\ell \).

Firstly, it is necessary to choose a helpful reference geometry. This will be taken as two parallel fixed edges, possibly the bars, with the grain or longitudinal axis of the soundboard at a slight angle \( \theta \) to the perpendicular between the edges. For analysing a soundboard, this is equivalent to a grain along the centre line, and slightly angled parallel bars, or bridge.

The new effective longitudinal modulus between the edges might be expected to behave as

\[
E_{l_{\text{new}}} = E_\ell \cos\theta + E_t \sin\theta \quad \text{or} \quad E_{l_{\text{new}}} = E_\ell + E_t \theta \quad \text{for small angles}
\]

(Sines have been suppressed in all the vibration theory, but are necessary below.) With the typical value of 14GPa for \( E_\ell \), 0.5GPa for \( E_t \), and \( \theta = 3° \) (or \( 3\pi/180 \)) for \( \theta \), the new \( E_{l_{\text{new}}} \) is 14.026GPa. This produces a negligible change in a longitudinal frequency \( f_\ell \).

Next, the new effective transverse modulus would behave as

\[
E_{t_{\text{new}}} = E_t + E_\ell \theta
\]

The new \( E_{t_{\text{new}}} \) is 1.23GPa, which is a large relative and absolute change in the transverse modulus. This increase by a large factor of 2.46 would increase a transverse frequency by a factor of 1.57 (or \( \sqrt{2.46} \)). The effect arises because a small fraction of a large quantity is added to a small quantity, unlike the longitudinal case. This is an exception to more usual instances of similar perpendicular values and larger angles. Sometimes, angles produce interesting phenomena.

The second point is the finding of increases in transverse frequencies \( f_t \) for panels, caused by the bridge and treble bar. These moderate increases usefully produced considerable spacings between the resultant resonances just above the basic longitudinal \( f_\ell \) of about 445Hz for a typical 10cm panel. This prompted a reassessment of the first mechanism for the elastic moduli of angled panels. The new transverse frequency \( f_t \) for bars angled at 3° would be increased from 9.2 to 14.5Hz. The new resultants when combined with the initial unmodified \( f_t \) would be

\[
\begin{align*}
    f_0 &= 445, 449, 464, 502, 574, 686, 838, \ldots \text{Hz} \\
    f &= 445, 447, 453, 469, 501, 555, 633, \ldots \text{Hz}
\end{align*}
\]

This gives appreciable spacings in the spectrum of resonances, compared with the initial values in paper 4 and in section 6 above:

The first five modes are spaced over 130 rather than 50Hz. This is comparable to the effect found above for an unangled bridge or a treble bar, which also usefully lowered the basic \( f_\ell \). Angling the main bars, or a bridge, can therefore provide similar effect, but only for the \( f_t \).

Historically, makers might have discovered these mechanisms from an apparent increase of stiffness when a sheet of pine is flexed at a slight angle, and then a correlation with the acoustics. The effect would be a neat way of decreasing the relative longitudinal and transverse stiffnesses and frequencies, for a given piece of pine. The natural ratio of moduli is about 28, and it has been shown above how a lower value may be desirable for lowering the longitudinal resonances of panels together with increasing the transverse ones. The use of angling would allow independent changes in these two modes. For example, a basic \( f_\ell \) could be lowered by
reducing the soundboard thickness $s$, and then $f_t$ could be increased by angling to give well-spaced resultant resonances. Perhaps also, as a correcting measure, a piece of pine judged as too flexible might be given angled bars. Angling would not be essential on all lutes, which is consistent with its appearance on only a few old instruments.

Also, in section 7 an angled main bridge bar gave a length $w_s$ of the lower panel that increased from the treble side towards the bass side. This lowered an effective longitudinal $f_t$ on the bass side but did not change the transverse $f_t$. The relative angles between main bars may therefore be used for making changes more independently in $f_t$ and $f_l$ than is possible with only the minor bars and an unangled bridge. These effects of angled main bars would apply to angled bridges, but less strongly and extensively, and also to the upper panels. The ancient makers therefore had a wide range of potential designs and adjustments by using angles.

14. Practical combinations of angles

There are no less than nine types of combination of angles for the bridge and bridge bar. Each type has a different taper of the lower panel and the intervening width of sensitive free soundboard. The position of the bridge within the panel would also affect the relative strength of resonances, as in section 3. It is best to indicate briefly which geometries seem acoustically useful, and give some examples of old lutes. Any complete treatment beyond that given above would require precise geometric details. Several photographs and drawings in Ref 4 show angled main bars, but it is difficult to know which are original and angled deliberately. Ref 4 and photographs in many other books, such as Ref 5, show bridges on baroque lutes angled in all three ways. Ref 6 indicates that over half the known surviving lutes have bridges angled to give shorter bass strings.

Much of the analysis has been based on the drawing in Ref 1. The bridge is without angle, and the angled bridge bar lengthens the lower panel and the free strip towards the bass side. This may lower, space out, and enhance resonances, especially for the bass side. A famous chitarrone by Tiefenbrucker in the RCM museum has all the main bars and the bridge at the same angle of about 3° (eg Ref 6). The author has confirmed that the soundboard grain is in fact at 87° to the bars, so this would be a firm example. The angle of the bridge and bridge bar away from the tail would lower $f_t$ for the bass side of the panel, and the use of a long J bass bar would also favour strong resonances for the long bass strings. The angling effect in section 13 could also be ensured for the upper panels, even though they are less strongly driven. These two cases were useful, and the less common related case with a bridge that lengthens the bass strings could give an even lower, stronger bass.

For an unangled bridge bar, which is likely on most lutes of any period, a parallel bridge is the normal, and modern, useful case examined above. A bridge that lengthens the basses might also improve the bass response. However, a more common bridge that shortens the basses and narrows the free strip on the bass side might raise the panel resonances and brighten the bass, as outlined above for baroque lutes. These three cases do not have a longer panel on the bass side. The remaining possibility of a bridge bar which shortens the bass side of the lower panel does not seem useful, particularly for the two bridge options that can narrow the free strip on the bass side. No real examples are available, but these cases might provide more stability, and brightness for the extra basses.

The present analyses of strongly angled minor bars and slightly angled main bars imply complications for treating instruments such as modern guitars. Here the detailed motion may involve twisting, whereas the normal unangled lute bars vibrate in the plane of the bar depth. A start might be made for fanned main bars by taking components of a simple bending motion. This is a further example of the simplifying scientific nature of the lute.
15. Detailed theory for angles

The basic effect in section 13 is physically sound, but details of the variation with angle \( \theta \) show that a more rigorous treatment is required for self-consistency. The forces acting on a non-isotropic angled plate need to be expressed in terms of the stresses in two perpendicular directions. The relations are found to be

\[
F_y = \varepsilon_y(E_t - E_i)\sin\theta\cos\theta + \varepsilon_y(E_t\cos^2\theta + E_i\sin^2\theta)
\]

\[
F_x = \varepsilon_x(E_t\sin^2\theta + E_i\cos^2\theta) + \varepsilon_x(E_t - E_i)\sin\theta\cos\theta
\]

where the x-axis is along the upper and lower edges, or the bars. The simple relations for \( E_{th} \) and \( E_{m} \) give an incorrect variation with \( \theta \) for an isotropic plate for which \( E_t \) and \( E_i \) are equal. These new relations correctly give \( F_y = \varepsilon_yE_t \) and \( F_x = \varepsilon_xE_i \).

The new forces and bending moments produce cross-terms in the partial differential equations, which need a new solution. The full expressions for general angles are complicated, but for the present case of a small angle and a small ratio \( E_t/E_i \), the resultant resonances were eventually found as

\[
f^2 = (\frac{(E_t/w_s^4)}{4} + (m^4E_t/l_b^4) + (2m^2E_t\theta/w_s^2 l_b^2)) \pi^2 s^2/48\rho
\]

This is for the first longitudinal mode with \( n = 1 \). The first two terms are the earlier standard form, and the newly derived third term expresses more correctly the dependence on angle, with \( f \) varying as \( m \) rather than \( m^2 \).

The effect of angle in raising the transverse frequencies can be expressed by the factor

\[
\sqrt{1 + (E_t/E_i) (l_b/w_s)^2 (\theta/m^2)}
\]

The new resonances are now

\[
f_0 = 447, 456, 474, 506, 554, 628, 718, \ldots \text{Hz}
\]

For the lower modes this actually gives a larger effect of angle and a greater spacing than the approximate method.

The puzzle of angled bars and bridges can now be seen to have a practical explanation, which also depends on fundamental theory. There is a physical analogy between an angled grain and the taper of strings. They both translate initially small variations into sizeable effects, the former on resonances and the latter on tuning. The two effects would also have been related in practice by the angled bridges. Small angles have perplexed our modern orderly perpendicular minds, but the original makers now appear to have used them in a range of mutual uses for strings and soundboard acoustics. It was initially thought that the angles on some old lutes might lead to general insight for analysing soundboards and strings. However, the understanding and theory needed to develop independently, and explanations of both practical effects may be a final bonus.

The next communication moves from these fine details of design to wide ranging effects with different sizes of lutes.

References
1. S. Barber: Lute Society drawing and notes for an early six course lute.
LUTE SIZES AND OPTIMUM DESIGN (7)

This final communication brings together all the preceding analyses of strings, resonances and stability, to see how a complete lute design and different sizes of lute depend on physical principles. The various processes were seen to depend differently on size and shape, and this has implications for the overall functioning of a single lute; for designing a family of lutes; for related instruments; and for the lute’s evolution.

1. Introduction
Old methods, and some modern instructions, for making lutes begin with a set of proportions for drawing the outline of a soundboard, the positions for a rose, bridge, bars, and angled necks. This reflects ancient ideal proportions and geometry for sculptures and buildings, which changed over time. The method uses few measurements, and has little relation to modern engineering drawing. These proportions may seem to imply some perfect operation, but it is unlikely that they correspond with a solution of some overall physics problem.

A useful aim of any fundamental study of a practical problem is to predict optimum combinations of variables such as sizes, shapes, properties, frequencies, etc. For lutes this was not possible previously, since there was no theory for the acoustics, the stability, and some details of strings. However, the importance of the original gut strings had been realized. These three topics can now be united to see how a lute may be optimized, or even designed from first principles. This cannot appear out of a vacuum, and account must be taken of the basic form, original materials, musical purpose, human physical abilities, and the lute’s history.

2. Lute strings
The original gut strings can be taken as a starting point. The frequency for any string of sounding length \( l \) is

\[
f = \sqrt{\frac{T}{\rho}} \left( \frac{2}{l} \right)\]

where \( \sigma \) is the stress and \( \rho \) the density of gut, as in paper 2. The maximum possible stress is the breaking stress \( \sigma_b \) of the gut, so that a maximum frequency is determined by the string length, or vice versa, with allowance for a safety factor. This relation applies to a treble string, since lower strings have lower stresses. Now points of physics and material properties combine with human preferences. Possible string lengths would need to be convenient for holding a lute and the hand action, particularly the stretch and agility on the frets. This led to a range of practical string lengths, with a range of musically useful maximum pitches for the top strings, described by the above equation. Next, the thinnest possible top strings, of order 0.4mm in diameter were found to give a humanly comfortable tension of about 3kg. This tension \( T \) and diameter \( d \) are related to \( \sigma_b \), by \( T \leq \pi d^2 \sigma_b / 4 \). This second relation has also been a fortunate result of human touch, in both \( T \) and \( d \), and the material property \( \sigma_b \).

All the old gut strings below the top string needed good elasticity, and the thickness of the lowest bass could not be made unnecessarily large, in order to reduce all the problems of stretch sharpening, inharmonicity and dullness discussed in paper 2. Even for the thinner higher strings, these defects reduce the prominence of many harmonic overtones, which appear to have been crucial for the brilliant yet sweet sound that is, and probably was, considered to be desirable in a lute. Therefore, the original lutes of all sizes would have had two reasons, in the treble and the bass, for using the thinnest possible gut strings. This ancient happy congruence of physics, natural materials, human hands and musical pitch seems generally accepted.
In the modern revival of the lute, much stronger string materials became available. However, the above measures of tension and diameter have remained comfortable. Much thinner top strings feel odd, and much higher tensions would require a more muscular technique. Lower tensions give a reduced resistance or feel, with a tendency to produce a non-linear confused vibration, and strings slapping the frets. The remaining possibility is a thicker top string, but this would sound duller and lack the higher overtones possible for thinner strings. The old limitations on the thickness of the bass strings would not be so severe with modern metal-wound strings, but these would be needed on more courses.

These similar requirements for gut and synthetic modern strings were arrived at gradually. Initially, modern lutes were heavier than the originals and used thicker, higher tension guitarlike strings. Later on, the lutes were based closely on the originals, which have survived in various sizes, but without any strings or specified pitches. Their original pitches or frequencies have been surmised from music, writings, pictures, and by trying various synthetic and gut strings. The above outline has tried to present the underlying ‘biomechanical’ relations.

In practice, the broadly accepted series of pitches and string lengths is: a bass D lute 80cm, F lute 67cm, G lute 60cm, and descant A lute 54cm. Rarer extremes are a low C 90cm, and a high c 45cm. The pitches, f, of these lutes are simply inversely proportional string length l, as a result of all sizes of lute using the same set of thicknesses, and changing only the string length. This is very different from other families of instruments, such as the violin having much thinner strings than a cello, with no simple scalings of string length with pitch. However, even for lutes there can be variations, such as G lutes working well with string lengths almost equal to those in A or F. (Any added complication of historical pitch standards is not crucial here. If the pitch of a solo lute is given as \( a = 390\text{Hz} \), for example, this indicates an F or E lute in the present context, not a smaller G lute at a different standard, with a drastic change in tension or string gauges.)

The next two sections show how the size and shape of a lute, the soundboard, bars and bridge, and the resulting resonances, may be determined by the different string lengths at about the same tension. The large overall range of size and pitch is covered, and also the smaller variations mentioned above. The acoustics and stability analysis in papers 4 to 6 are used with the above account of strings. In addition to the basic pitches of strings, longer and thinner strings will be able to sustain relatively more harmonics. Strings on lower pitch lutes, and also slightly longer strings for a given pitch of lute, therefore have the potential to sound richer than the shorter strings on smaller lutes.

As a brief guide to the long analysis in sections 2 to 7, the successive stages can be listed as: pitch; leading to string length and instrument size; plus stability leading to restricted choices for other dimensions; leading to the acoustics; then forcing by strings from the previous stages. In principle, this provides a complete and consistent explanation for lute design, in effectively a single large equation.

3. Lute stability and size
The following analysis primarily compares lutes with the same number of strings, such as basic six or seven course designs. However, at various points it can be seen how the conditions for a particular size could permit an increased tension, or extra courses, or need more strengthening. The acoustics work in paper 4 showed that for satisfactorily low resonances, a lute body needs a large volume and width \( A \). A large body length means that the length of neck should be short. Only eight frets on the neck were found to be musically sufficient, which is fewer than for many other similar instruments. Account must also be taken of the evolution of an oval body shape from a natural form, and its constructional purpose; and also the reasons for the position of a bridge given in papers 5 and 6. The stability theory showed how the body length \( l \), and the relation for the permitted degree of bending \( k \), determined compatible values for the other main
dimensions of a lute. These are: lute width or main bar length \( l_b \), bar depth \( b \), bar width \( w_b \), bar number \( n_b \), action height \( a \), and total tension \( T \).

It is useful to include \( n_b \) in \( w_b \), and scale all the dimensions in the relation for the bending \( k \):

\[
\frac{k}{l} = \frac{(3/2)(T/E)}{(l_b/l)^3(a/l)} \left( \frac{(b/l)^3(w_b/l)}{l^2} \right)
\]

The main possibilities for scaling these six variables were indicated in paper 5. A systematic approach is long, but preferable to choosing a few isolated cases. The influence of so many different length scales has required detailed elastic theory and could not follow from dimensional analysis, or without a wide range of experiments. Some of the predictions will correspond with normal experience, but others are surprising.

As a starting point, \( l_b, b, w_b, \) and \( a \) can all be simply proportional to lute size \( l \), so that the same relative bending \( (k/l) \) varies as \( T/l^2 \). This indicates, for example, that a D lute could have a total string tension 1.8 (or \((4/3)^2\)) times that of a G lute, whereas an A lute could take only 80% of the tension. The comparable factors for the extreme C and c lutes are 2.25 and 0.56, or an overall factor of 4. However, from the above treatment of strings, tension is kept broadly constant for all sizes of lute. Practically significant departures are used in modern lutes, such as slightly higher tensions and thicker strings for larger lutes, and for an individual lute a gradual increase of tension across the courses from bass to treble. These are small in the present context of size and design, and are considered later.

Now with a constant tension, the four lengths, \( l_b, b, w_b, \) and \( a \), can be varied. There are no less than 32 possible types of variation, but this number can be reduced if the total mass of the bars \( M \), or \( \rho w_b l_b \), is kept constant for each lute size \( l \). The relative bending becomes

\[
\frac{k}{l} = \frac{(3/2)(T/E)}{(l_b/l)^3(a/l)} \left( \frac{(b/l)^3(M/\rho l)}{l^2} \right)
\]

The mass of bars \( M \) can be kept proportional to \( l^3 \), even though it may be only a small fraction of the total lute mass. This allows the main effects of three lengths, \( l_b, b \) and \( a \), to be analysed with 12 cases. There are no turning points, but just balancing variations, and all the cases can be listed. The main practical interests are relatively lower bar depths \( b \), and larger lute widths \( l_b \), for larger lute sizes \( l \). Some of the variations, such as an \( l_b \) that does not increase at all with lute size \( l \), will have little acoustic interest. The resulting variations in \( w_b \) can be found from the inverse of \( b/l \). The six remaining useful cases are:

<table>
<thead>
<tr>
<th>Lute size ( l )</th>
<th>2.0</th>
<th>1.5</th>
<th>1.33</th>
<th>1.12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l_b )</td>
<td>2.8</td>
<td>1.8</td>
<td>1.5</td>
<td>1.19</td>
</tr>
<tr>
<td>( b )</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>( T )</td>
<td>8.0</td>
<td>3.4</td>
<td>2.4</td>
<td>1.4</td>
</tr>
<tr>
<td>( l_b )</td>
<td>3.4</td>
<td>2.0</td>
<td>1.65</td>
<td>1.22</td>
</tr>
<tr>
<td>( b )</td>
<td>0.7</td>
<td>0.82</td>
<td>0.87</td>
<td>0.94</td>
</tr>
<tr>
<td>( l_b )</td>
<td>2.4</td>
<td>1.7</td>
<td>1.4</td>
<td>1.15</td>
</tr>
</tbody>
</table>

All the potential uses can be seen from this table of factors for various relative lute sizes. As a single complete example, a factor of 1.33 in \( l \) can compare a lute in D with one in G. Without analysis, or much previous testing, or a proven design to copy, all dimensions might be scaled simply by a factor of 1.33.

Case 1 shows that the width can be increased beyond a simple factor of 1.33 by a further 1.13 (1.5/1.33) or 13%.

Case 4 allows a greater increase of 24% in \( l_b \), just by keeping \( a \) fixed.

Case 6 still shows a 5% increase in \( l_b \), even if both \( b \) and \( a \) are fixed.

Case 2 allows the bar depth \( b \) to be kept the same, if \( l_b \) and \( a \) just scale.

Case 5 can even enable a 13% reduction in \( b \), just by fixing \( a \).
Case 3 shows that if the main dimensions $b$ and $l_0$ just scale, then $a$ could increase 2.4 times. This is not useful, so the tension could be increased by the same large factor. The stability could also be improved by reducing $k$. There is no interest in cases with larger increases in $a$.

The other scaling factors can compare lutes in $F$ and $G$ ($l \times 1.12$) or $A$ with $G$; $D$ with $A$ (1.5), the extreme range $C$ with $c$ (2.0); and $D$ with $F$ would need 1.2. The actual dimensions of barring were treated in paper 5 for stability, and in papers 4 and 6 for the position of resonances within the range of a lute.

The above example illustrates the physical principles behind stability. It shows the useful but initially puzzling possibility of making longer lower pitched lutes relatively wider and more lightly barred. This in turn can produce relatively lower resonances in larger lutes, or the converse for smaller lutes, as explained in the earlier papers and below. Historical examples of barring on different pitched lutes (Ref 1) appear to show similar bar sizes for a wide range of lute sizes, but the handwritten sizes are not clear enough for detailed calculations. The Frei lute, in papers 5 and 6, which has been a popular model for many $F$ lutes and baroque lutes, has bar sizes similar to many $G$ lutes, and also the tension of extra strings. The practicality of relatively greater widths for bigger lutes would be limited by bulkiness, just as much greater string lengths also placed an upper limit on size.

So far the analysis has not included the mass of the soundboard and body. Their surface area is accounted for by the sizes $l$ and $b_0$, with some variation arising from the detailed shape. The remaining factor is the thickness of these parts. This did not occur explicitly in the stability theory, where it was only necessary for the soundboard to be glued strongly to the bars, then to the body sides, and also be able to withstand the forces and moments on the bridge. This just required a minimal practical working thickness. It could be 3mm but this is unnecessarily thick, and it would raise panel resonances and be harder to force with the strings. Conversely, 1mm would be fragile, and give low, reedy, non-linear panel resonances. Hence 2mm or slightly less can give adequate stability and resonances. A measure of the surface area might be $\frac{1}{2}\pi b_0 l$ for the belly, and about twice this for the body. Taking a common thickness $s$, the mass would be proportional to $\frac{1}{2}\pi b_0 l s$, compared with say $6bw b_0$ for the bars. The bar mass as a fraction of the total is about $(b w b_0 / 3 s)$, which is only 2% using previous typical values. The simple scaling of bar mass as $l^3$ might be reduced to $l^2$ to match the body mass, which would give a slightly lower advantage to larger lutes. The increases of $b_0$ would be reduced by $l^{1/4}$, those of $a$ by $l$, and the decreases of $b$ reduced by $l^{1/2}$. However, the very small proportion of bar mass would justify much larger increases with size. Hence, if the analysis showed an acoustically useful case, the advantages for larger lutes predicted above could be increased even further.

The next section concentrates on the acoustics of different sized lutes in relation to the analyses of stability and strings. The stability relations will also be seen to have a natural extension to differently shaped lutes. In section 5 the much fatter short-necked lutes such as the oud will provide an instructive comparison with the European lute. The analyses are also applied to early guitars and vihuelas in section 6.

4. Lute acoustics, stability and strings

In the initial acoustic predictions of paper 4, it was found that if certain key dimensions were kept constant then larger lutes would have relatively lower resonances. In the stability analysis a further prediction was that larger lutes may have dimensions that could provide three advantages together: stability at least as good as smaller lutes; lower resonances relative to their pitch; and the same or lower mass relative to their size. The second point could produce a more sonorous lute, and the last point would improve the responsiveness. These effects can be anticipated from the previous relations, and a final detailed overall analysis is necessary. This will also form an extended conclusion for much of the present research. Practical examples and implications will be included throughout the analysis, rather than in another lengthy section.
The initial reference point is that the pitch $f$ of different sizes of lute generally varies as $1/l$, as explained in section 2. Different pitches or sizes of lute can be compared by the relative values of $l$ listed in section 3.

Starting from the lowest resonances, in paper 4 the possible vibrations of the body shell, and also the entire length of lute, could have a frequency $f$ proportional to $e/i^2$, where $e$ is some effective thickness. If $e$ is scaled simply as $l$ then resonances will vary normally as $1/l$. If $e$ remains about the same for larger lutes, which may be possible for soundboards, ribs and necks, these frequencies would scale as $1/i^2$. This would lower resonances by a further factor of $1/l$. A transverse vibration of the shell would vary as $e/l^2b$, and further lowering will be possible if stability allows $h_b$ to increase more strongly than $l$, as treated below.

Next, the Helmholtz resonance of the enclosed air had a frequency $f_h$ that varied as $\sqrt{(A/nV)}$, where $V$ is the enclosed volume of air, and the neck has a cross section area $A$ with an effective length $n$. If all dimensions scale with size $l$, then $f_h$ would vary normally as $1/l$. To examine possible departures, in paper 4 $(A/n)$ varied as the rose diameter $d$. If the lute body is treated as an ellipsoid with axes $l$, $h_b$ and $l_d$, where $l_d$ is twice the depth $d$, its volume is $\pi d_d l_d/12$. The frequency $f_h$ now depends on $\sqrt{(d/l_d)}$, and several variations with size $l$ are possible. Paper 4 noted that if $d$ remained fixed while all the lengths scaled then there is a further lowering of $f_h$ as $1/\sqrt{l}$. If a larger width $h_b$ is allowed by the stability requirements, $f_h$ decreases by an additional $\sqrt{(l/h_b)}$. If $l_d$ also behaves like $h_b$, the total decrease of $f_h$ relative to simple scaling could be a factor of $((l/h_b)/\sqrt{l})$. As an example, lowering a tone from a G to an F lute requires $l$ to increase by a factor of 1.12. If $(l_b/l)$ also gained an extra increase of 1.22 from stability case 4, then the further relative lowering of $f_h$ could be $(1/1.29)$. An $f_h$ of 130Hz for the G lute would scale simply for an F lute to 116Hz, with a predicted possible decrease to 90Hz. This would move the Helmholtz region down to the sixth course from the fifth. This may seem an extreme example, but the lesser increase of $(l_b/l)$ in case 6 still gives 95Hz.

Although relatively wider larger lutes could have relatively deeper resonances, there would be an increase in bulkiness for the bass C and D lutes. These are already near the limit for holding, and hand size, and some players have a problem even with E and F lutes. Original lutes tend to show a characteristic belly outline of roughly constant proportion with small variations of perhaps less than 5% in $(l_b/l)$. This holds for the early pear-shapes and the later more plum-shaped form, with similar variations in baroque lutes. Some of the visual contrasts are in the shoulder near the neck, and not the effective $(l_b/l)$. Instead the early inventor-makers appear to have kept the bar depths $b$ and widths $w_b$ more constant for larger lutes (Ref 1), which is allowed by stability in cases 2 and 5 above. The option of much wider lutes, in cases 1 and 4 was not taken, but case 6 would allow a combination of both options. If wider large lutes were impractical or acoustically undesirable, why were smaller lutes, such as those in G, not made much wider. In fact these were made and called ouds: the ancestors of European lutes. The stability and acoustics of the oud, which may be termed fig-shaped, are examined later.

The most interesting aspect for these analyses is the behaviour of the low overall modes of the belly, and especially the transverse frequency given in paper 4 as

$$f_t = (b/l_b^2) (1/4\pi\sqrt{3}) m^2 \sqrt{E/p(1+sw)(bw_b)}$$

Here the main effects of scaling are in the first term $(b/l_b^2)$. If $b$ and $l_b$ are scaled simply as $l$, then $f_t$ would vary normally as $1/l$. If $b$ is allowed to increase less than $l$, stay constant or even decrease slightly, then $f_t$ can decrease more strongly for larger lower lutes. This can be made clearer if the lute width $h_b$ and action height $a$, are scaled initially as $l$. The stability relation for $(k/l)$ now requires $(b^2w_b/l^2)$ to be a constant, and the bar mass $M$ varies as $bw_b$. This means that the bar depth $b$ varies as $\sqrt{(l^3/M)}$, and bar width $w_b$ as $\sqrt{(M^3/l^5)}$. The results can be seen in a table, where for a constant $(k/l)$ any two of $b$, $w_b$ and $M$ determines the third:
b varies as $b^{1/2}$

$w_b$ varies as $1/l$

$M$ varies as $M^{1/2}$

<table>
<thead>
<tr>
<th>Column</th>
<th>b</th>
<th>$w_b$</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$l^{1/2}$</td>
<td>$1/l$</td>
<td>const</td>
</tr>
<tr>
<td>2</td>
<td>$l$</td>
<td>$l^{1/2}$</td>
<td>$1/l$</td>
</tr>
<tr>
<td>3</td>
<td>$1/l^{1/2}$</td>
<td>const</td>
<td>$1/l$</td>
</tr>
<tr>
<td>4</td>
<td>$1/l$</td>
<td>$l^{1/2}$</td>
<td>$l^{1/2}$</td>
</tr>
<tr>
<td>5</td>
<td>$l^{1/2}$</td>
<td>$l^{1/2}$</td>
<td>$l^{1/2}$</td>
</tr>
<tr>
<td>6</td>
<td>$l$</td>
<td>$l$</td>
<td>$l$</td>
</tr>
</tbody>
</table>

Column 2, in which b scales simply, allows an inverse decrease of $w_b$, for only slight increases of mass. The last two effects may seem useful but the first does not give extra lowering in $f_t$. If, in addition, $w_b$ were increased simply as $l$ and the mass as $l^3$, then $(k/l)$ would be reduced, giving excessive stability, or the tension could be increased. This would be simple scaling, or case 3 above. This limit and lesser intermediate cases would be available for larger lutes with more tension and courses. The underlined values are of more acoustic interest.

Column 3 gives some relative reduction of b, and allows a fixed $w_b$, for a modest mass increase. Column 4 allows b to be fixed for all lute sizes, and a normal scaling of mass, but starts to need strong increases in bar width. Column 5 allows that absolute reductions in b for larger lutes begin to need large increases in bar width and mass, giving lower forcing and slower response.

The case of a constant $w_b$, as in column three, has practical interest in addition to its use for stability and acoustics. A thickness of pine sufficient to glue a bar in good contact with the soundboard would need to be greater than the soundboard thickness of 2mm. A reasonable minimum might be 3mm, while more than 5mm would be unnecessary, and an average of 4mm was taken in paper 4. In column four, the other useful variation of $w_b$ as $\sqrt{l}$, or a factor of 1.4 over the entire range of lute sizes, is within the variation of 3 to 5mm. The same factor would also cover a range of bar depths from 12 to 17mm. This corresponds with the design of real lutes, whereas a simply scaled doubling of b is not likely. During the practical construction of a lute, the lengths b and $l_b$ would be fixed, then $w_b$ could be chosen and even replaced, and next the adjustment of bar depth b by planing down would have the most flexibility and sensitivity.
Now the overall longitudinal frequency needs attention, and paper 4 gave
\[ f_l = \left( \frac{s}{I^2} \right) \left( \frac{1}{4\pi \sqrt{3}} \right) m^2 \sqrt{\left( \frac{E_f}{\rho (1+bw/b_s)} \right)} \]
As before, if the soundboard thickness \( s \) remains constant the main variation of \( f_l \) is \( 1/I^2 \), and if \( s \) increases simply as \( l \) the variation has normal scaling as \( 1/l \). The inertial term is again constant, for the variations in column four. For the alternatives in column three, this term would tend to \( l^{1/6} \), with \( f_l \) as \( 1/l^{11/6} \). The relative lowering of the smaller \( f_l \) frequencies is therefore similar to that for the larger \( f_l \), for both of these useful cases. This would be satisfactory for any uniform lowering of the resultant resonances formed from \( f_l \) and \( f_t \). In contrast, column five with a fixed \( b \), and \( w/2 \) increasing as \( l^2 \), would give an inertial term that leads to much stronger decreases of \( f_l \). Lowered resultants would be bunched up near the lower Helmholtz region, giving a strong bass response, and leaving a gap in the middle range of resonances. All this suggests a benefit of making larger lutes with main bars that have a depth and width which increase slowly with size as in columns 3 and 4, and where possible a slight extra increase in lute width.

As noted above, the lowest lutes in C and D with a naturally stronger bass, are not solo instruments. Similarly, the natural relatively weaker bass of the highest lutes in A and C, and their high pitch, is not for general solo use. They are reserved for special effects, requiring greater agility, which may even become limited by small spacings between frets. The useful gains of sonority for solos or duets would be in the middle range, between say E and F#. These sizes were also used later for baroque lutes, where the number of strings was doubled.

For lutes played in a consort there may be a contrasting situation. Here, the range of three octaves for a single lute can be extended by only one more octave, which is small compared with other groups of other instruments. However, the relatively low resonances for the lower lutes, and the prominent higher frequencies for the higher lutes can be satisfying, and an increase of these extreme properties could enhance the effect. The simple scaling of string lengths and lute sizes with pitch also tends to produce similar timbres for all the various sizes of renaissance lute. A consort can therefore sound like a single enormous lute, unlike a string quartet or wind consort where each instrument is identifiable.

Finally, the higher resonances of separate panels are based on a longitudinal \( f_l \) that varies as \( (s/w_s^2) \), or \( (s/l^2) \), as for the overall frequencies. If \( s \) stays constant the dependency is \( 1/l^2 \), and if \( s \) increases as \( l \) the variation has the normal scaling \( 1/l \). The transverse \( f_t \) varies as \( (s/l_0^2) \). As noted above, the lute width \( l_0 \) was not increased in practice much more strongly than \( l \), but any small variations could be found from a possible joint variation in \( (b/l_0^2) \), discussed with the more important low \( f_t \). The very low \( f_t \) for the body shell, at the beginning of this section, would have similar variations. Keeping the soundboard thickness fixed for larger lutes would cause a further relative reduction of the resultant higher resonances as \( 1/l \). The potential for longer strings on lower pitch lutes to have relatively more high harmonics and sound richer than the shorter strings on smaller lutes was mentioned in section 2. This might not be reinforced well by greatly lowered soundboard resonances. There would be some compensation in a quicker response, which would be useful for larger lutes. In practice, the variation of thickness \( s \) may be midway, and this was also deduced for the vihuelas cited in paper 4. A possible variation could be \( \sqrt{l} \), which was useful for bar depth. A new criterion for soundboards is suggested in section 7.

This completes the combined analysis of strings, stability and acoustics, which indicates how an individual lute and different sizes of lute can be designed from physical principles. Some major practical consequences have also illustrated the theory. The degree to which real lutes fit the theory may depend on several factors: the correctness of the theory, whether principles were completely discovered or followed, and whether departures could have a neutral effect. Many of the tight interdependences have shown that departures can degrade the performance. Next, three other related instruments are considered, before examining how the driving of lutes by the strings may vary with design and size.
The oud is mentioned only rarely in relation to the lute, even though it was the original model for the European lute, and still survives. It came to attention above as an example of a much wider moderately sized lute. The acoustics suggested possible advantages of lower resonances for wider lutes; and the stability theory indicated how this could be achieved. Both these aspects can be treated, and show why the oud is distinctive from a lute. A fine oud from 19th century Damascus is described in Ref 2. This may be better than modern ouds for a comparison with old lutes.

The string length of six double courses is 61.4cm, and the larger dimensions have been estimated from the photographs. Fortunately, the smaller dimensions were provided. The fig-shaped body is about 51cm long and 37cm wide. The ratio \((l/h_b)\) is 1.35, or 1.65 based on string length. This contrasts with 1.5 and 2.0 for the typical G lute, for which a width of even 32cm would be large. The next difference is the far greater number of main bars below the rose: five in the oud compared with three or even two for a typical lute. The bars are less deep and considerably wider for the oud, and they have an equal spacing of about 5cm. These features of the oud were mentioned, in papers 4 to 6, as general possibilities for the lute. Bars with a squarer section in the earliest lutes may have originated in the oud.

The stability and resonances can be analysed with the relations devised for the lute. Results that previously needed three papers to describe can now be presented in a single table.

| String body neck width Bar depth width number Stability \(f_t\) \(f_b\) |
|---|---|---|---|---|---|---|---|---|---|
| l (cm) | \(h_b\) | b (mm) | \(w_b\) | \(n_b\) | \(b/h_b\)^3 \(b/h_b\)^3 \(w_b n_b\) | \(b/l_b^2\) | \(\sqrt{d/V}\) |
| Oud | 61 | 50 | 20 | 37 | 13 | 6 | 5 | 4.3 | 12.9 | 9.5 | 13.5 |
| G lute | 60 | 45 | 25 | 30 | 15 | 4 | '2\(1/2\)' | 12.5 | 12.5 | 16.7 | 14.9 |

The relation for bending \(k\) in section 3 shows that a similar level of stability for the oud, and a lute of the same size \(l\), requires a similar value of \((b/h_b)^3 w_b n_b\). It is seen that \((b/h_b)^3\) for a single oud bar is only about a third of the value for a lute bar. Part of the difference is due to the greater width of the lute, and a smaller part to the lower bar depth. An oud bar is 50% wider, so that an oud still needs twice as many bars for stability. The five bars on the oud therefore give the same measure of stability as 2 to 3 bars on the lute. (The arbitrary units in the last four columns give consistent comparisons.) These calculations assume similar total string tensions, which are likely. The height of the string holes on the bridge is 5mm, as for the lute.

The last two columns give measures of the low transverse frequency and the Helmholtz resonance. The value of \(f_t\) is almost halved for the oud due to a reduced \(b\) and an increased \(h_b\). The reduction of \(f_t\) from the added mass of soundboard is smaller for the oud, and like the 5cm panel used initially for the lute in paper 4. The final \(f_t\) for the oud would be about 65% that for the lute. The value of \(f_b\) is lowered only slightly because while the volume is much greater, the diameter of the carved rose is 12cm compared with about 9cm for a lute. A comparison is given using the smaller lute rose. The volumes are based on the ellipsoids in section 4.

The resulting sound of the oud would be a stronger deeper bass, with the \(f_t\) and \(f_b\) quite close. The higher resonances of panels based on \(f_t\) would be raised by a factor of about 4 because all the panel lengths \(w_s\), including those in the lower active area, are half the 10cm for the lute. Both these effects would lead to far fewer mid range resonances than for the lute. The soundboard thickness is about 2mm, similar to a lute. The large mass of lower bars is 3 times that for a typically barred lute and would be harder to drive and slower in response. The bridge is placed midway between the lowest two main bars, and there are no minor bars. This analysis is consistent with a type of lute suitable for playing rapid runs with a plectrum and dipping down to the lower strings, above a loud bass with occasional chords.
It can also be seen how the European lute was developed to provide graded resonances over the whole range, suitable for its new polyphonic music. A weaker bass and higher overall $f_i$ modes could use a narrower width, and hence fewer and narrower bars were needed for stability. This would also allow more mid-range resonances, blending into the higher resonances. The larger lower panel, from the absence of the lowest bar, would give a lower $f_i$ for the panel. This might soon have required local stiffening from a treble bar to give higher transverse $f_i$, for well-spaced resonances. The mysterious bass J bar may even have some origin in the lowest bar of the oud. With its more gentle refined sound, the lute eventually declined. The oud evolved into the modern versions with an open soundhole and two smaller holes, perhaps becoming even louder. Many of the much earlier long-necked type of lute, such as the saz and variants on ‘tanbur’, could be treated. Their small bodies and fewer strings would reduce the problem of balancing stability against acoustics, and lower strings would be used for a drone rather than a true bass. The only European case of a true long-necked lute is the imported colascione with only three strings, a small body and three bars.

6. Early guitar and vihuela

The early guitar and vihuela had only two main bars, below and above the rose. This poses an interesting problem for stability analysis. However, the bars cross the narrow waist that has a width $l_w$ of about 18cm, which is 70% of the lower width $l_b$ of about 25cm. The bar depth and width are similar to the lute values, so that if two shorter bars can give the stability of five longer bars then the required bending relation would be

$$\frac{2}{l_w^3} = \frac{5}{l_b^3}$$

or

$$l_w/l_b = (0.4)^{1/3} = 0.73$$

This is surprisingly close to the actual ratio for instruments, and a practical use of the strong dependence of stability on $(l_b/l_w)^3$. The lack of bars near the bridge, and supporting it, may be offset by the smaller lower width. In this lower area the thickness $s$ of the soundboard itself would be important for stability, and need to function like the bars of a lute. This would require another stability analysis, analogous to that for the bar depth $b$. This might reduce the scope for keeping $s$ constant for larger vihuelas. Also, the shallow vertical sides and flatter back are more rigid than a lute shell, even before attaching the soundboard.

As found in paper 4, the greatest difference from a lute or an oud is that the resonances do not depend significantly on the bar depth $b$, which just has to provide part of the stability and an upper edge for the lower panel. The central narrow waist evolved from violas, where it enabled bowing. Acoustically it cleverly provides a large lower panel of free soundboard, together with a large volume from the upper body.

Different sizes of early guitars and vihuelas are known from writings, pictures and music. Three further sizes of vihuela designed from a single surviving size were discussed in paper 4. In the acoustics, a Helmholtz resonance $f_i$ could vary as $\sqrt{d/t \times l_b}$, where the belly may still have an area of about $V \times l_b / l_d$, and $t$ is the depth of the body, which is close to a third of $l_d$ for a lute. This $f_i$ would scale as described above for lutes.

The analysis of the soundboard in paper 4 predicted a series of longitudinal $f_i$ for the single large panel below the bars. This just depends on $(s / \sqrt{s^2})$ or $(s / l^2)$, like the smaller lute panels. If $s$ were able to stay constant the frequencies would vary as $1/l^2$, and if $s$ increases as $l$ the variation is the normal $1/l$.

The series of transverse $f_i$ varied as $(s / l_b^2)$. For a lute, the width $l_b$ was not increased in practice more strongly than $l$, and the dependency lay between $1/l^2$ and normal scaling $1/l$. For a guitar or vihuela, the stability depends on $l_w$ for the waist. This might be varied independently of $l_b$, but many examples show the above factor of 0.7. The stability of the bridge, without nearby bars, would also depend on $l_b$ and $s$, and the associated bending would need to scale with $l$. The $f_i$ could vary like $l_b$ and give the same behaviour for the resultant resonances. All this shows that the stability and acoustics of guitars and vihuelas differ in principle from a lute or oud. However, the sound and music of the first two is more like the lute than the bolder oud.
7. Forcing by the strings

The initial thoughts on the importance of the bridge area included how the driving of a soundboard might vary for different designs and sizes of lute. For any type of vibrating system this could lead to assessing the output, an efficiency, and any optimizations. A usual method might calculate the kinetic energy of a vibrating string, and then the amount transferred to the soundboard and air. This would be needed for a wide range of strings, and lute designs. After the static stability had developed into a large subject, a longer treatment of vibration seemed forbidding. An alternative approach emerged out of the stability work. The final surprising result was found to be that the previous three topics of strings, stability and acoustics have already defined the system. This depends on two main aspects and is best explained with only a few equations.

The first point is that the vibration energy can be expressed solely in terms of the potential energy stored in a stretched elastic string, and also in a bent elastic plate or bar. This was referred to in Rayleigh’s method in paper 6, and it can also be seen in the equations for frequency. For example, the relation for a string in paper 2 gives \( \frac{1}{2}(\mu I)(4f_0^2) = Tp^2 / 2l \), where a small amplitude \( p \) has been included. The left side is the kinetic energy, \( \frac{1}{2}Mv^2 \). The stored potential energy can be seen from stretching a string at a tension \( T \) by an amount proportional to \( p^2 / l \), as seen in paper 2. Equivalently, the string is drawn sideways an amount \( p \) against a lateral force \( Tp / l \), as in paper 4. Similar relations apply later for the motion of the soundboard and bars. This indicates that the entire system of a lute in action can be analysed in terms of the tension of one or more strings, and also the amplitude and string length. This is easier than five direct acoustic variables, such as string diameter, length, density, frequency and amplitude, and then also the soundboard variables.

Secondly, the tension directly relates the acoustics of the strings and lute with the stability. The main interest is the amount or amplitude of vibration produced in the soundboard and bars by a vibrating string. This has effectively been treated already by the stability theory in paper 5, and an earlier estimate of about 0.04mm was found in paper 4 on the acoustics. The rest of the section analyses this in detail.

The lateral force needed to bend a bar by an amount \( y \) was \( 4E_b(b / l_b)^3w_b y \). The force applied by a string was \( pT_s / q \) for an initial downward displacement \( p \) at a distance \( q \) from the bridge. Similar relations could apply for sideways plucking. The tension in a single string is \( T_s \), and the total tension exerted by all the strings will be \( T_l \). The amplitude \( y_b \) for a low overall vibration involving the bridge bar is then

\[
y_b = \left( \frac{T_s}{4E_l} \right) \left( \frac{l_b}{b} \right)^3 \left( \frac{p}{qw_b} \right)
\]

This relation is similar to the previous one for the bending \( k \) produced by the total tension. The small amount of active bending relative to the static bending is

\[
y_b / k = \left( \frac{T_s}{6T_l} \right) \left( \frac{p}{q} \right) \left( \frac{l}{a} \right)
\]

Using all the previous typical values for \( b, w_b, l_b, p, q, T_l \) and \( E_i \) of 15, 4, 300, 5, 150mm, 3kg and 14GPa, the amplitude \( y_b \) is about 0.04mm. (The ingredients for stability and optimization laid unnoticed in paper 4.) This amplitude may be considerably greater than in tests on stiffer guitars and violins. From optical fringe patterns in the various references of paper 4, the amplitudes may be estimated at some distance from a non-string forcing, but no measurements were given. In contrast, the present analysis will give upper estimates for downwards plucking. The relative value of \( y_b / k \) is 0.05 for a single string. The value of \( k \) is about 0.9mm, as given in paper 5, and would be divided between say five bars. The static bending of one bar would be about 0.2mm, and the superimposed vibration for one course less than a half. This may suggest a further condition that a realistic forced bending should be considerably smaller than the static bending allowed in the stability.
For bending a panel of soundboard, a similar estimated amplitude is

\[ y_s = \left( \frac{T_s}{4E_t} \right) \left( \frac{w_s}{s} \right)^3 \left( \frac{p}{q} \frac{l_b}{l_t} \right) \]

This value is relative to a fixed bar. Since a string forces the bridge, the soundboard panel will be acted on directly. An amount of bending \( y_b \) is then transferred to the bar, and the total bending of the soundboard would be \( y_b + y_s \). This is like two springs connected in series.

The most important aspect of these amplitudes is that they are proportional to the amount of bending \( k \) in the stability analysis, which in turn is best made proportional to the lute size \( l \). If the connection between tension in acoustics and in stability seems unusual, then a simple bow harp is a clearer case. Amplitudes also increase as the relative tension \( \frac{T_s}{T_t} \) for each of the strings played. They also increase with the ratio \( \left( \frac{a}{a} \right) \) and the degree of plucking \( \left( \frac{p}{q} \right) \), which would be about constant for different sizes. In addition to its effect on stability, the action height \( a \) will place a strict limit on the amount of plucking \( p \). Sideways plucking is limited further by the small separation between lute strings, which will also vary as \( l \). The relative amount of plucking will have a maximum value, but with a range of lower values for expression. All this indicates that in principle the requirements of strings, the lute size, the resonances, and the stability determine fully the maximum allowed forcing of the lute resonances.

A further ratio that will be shown to be of great interest is

\[ y_s/y_b = \left( \frac{bw_s}{s} \frac{l_b}{l_t} \right)^3 \]

or the relative amplitudes for high panel resonances and the low overall modes. This depends on all the various dimensions of a bar and panel. The typical values above, and also 2mm for \( s \) and 10cm for \( w_s \), give a value of 0.2 for \( y_s/y_b \). This shows that the higher panel resonances are similar to but weaker than the overall resonances. It might be useful for a balanced spectrum of resonances if both types of mode had similar strengths. Some of the dimensions may be adjustable to improve the detailed forcing by the strings, in addition to determining the main acoustic and stability effects. The value of \( y_s/y_b \) can be increased by larger \( b \), \( w_b \), or a smaller \( l_b \), none of which would be useful for the earlier acoustics of a low transverse \( f_t \) and good stability.

The relation also involves soundboard thickness \( s \), and there may be a physical criterion here for suitable values of \( s \) in the important lower panel. There could be no earlier guide to the thickness \( s \) in the acoustics, except the approximate frequencies, and the idea that a soundboard which is too thin would have an uncontrolled reediness, or flutter. The stability analysis required only a nominal strength for joints. The ratio \( y_s/y_b \) is very sensitive to \( s \), and \( y_s \) becomes equal to \( y_b \) as \( s \) is decreased to 1.2mm. Remarkably, this is about the minimum found in the variations of thickness across good soundboards. The relation also shows that even thinner soundboards would begin to give weak low overall resonances, \( y_b \). A thinner soundboard will simply bend more, and move the bars less, but a physical relation to all the other factors needs theory. An average thickness for fine lutes may be about 1.6mm and this value, with lighter 12mm deep bars, was used in papers 4 and 6 for further calculations of characteristic resonances.

The equation for relative amplitudes of vibration has a close and unexpected resemblance to the treatment of the active area around the bridge in paper 5, section 2. A characteristic length \( l_c \) for this lower area was derived from the low overall frequencies \( f_t \) and \( f_b \), including the large effects of added mass. Combining the two relations gives

\[ y_s/y_b = \left( \frac{w_s}{l_c} \right)^3 \]

This indicates a need for a balance between lowering \( s \) in order to increase \( y_s/y_b \), without greatly shortening the active length \( l_c \) to much nearer the bridge bar at \( w_s \), or even lower. This would be a subtle practical purpose. If \( l_c \) is reduced a little from the previous 17 to 13cm, then \( y_s/y_b \) is 0.45, which may be sufficient, while \( s \) is 1.5mm, which is close to a realistic average value. These adjustments may therefore change the relative strengths of low and high resonances as well as their frequencies. Further detail may need to account for the bridge position on a panel.
It is also useful to see how the relative amplitudes scale with lute size. If lengths scale simply as \( l \) then \( y_s/y_b \) is unchanged, but if \( s \) stays constant the variation is \( (l/s)^2 \), which allows greater panel resonance in larger lutes. For both the variations of \( b \) as \( l^{2/3} \) and \( \sqrt{l} \), listed in the table of section 4, \( y_s/y_b \) varies as \( l/s \). This gives a smaller increase with size but could still allow a constant value of \( y_s/y_b \) for different sizes of lute, if \( s \) increased as \( l^{1/3} \). This requires a factor of 1.26 for a doubling of size \( l \), which can be compared with the \( \sqrt{l} \) variation in section 4, and also with the vihuelas in paper 4 and section 5. Local contouring of the thickness \( s \) across a soundboard could be treated, similarly to the minor bars in paper 6. All types of variation would be very sensitive around the bridge, and also for the panel above the bridge bar. Further comparisons are possible but the approximations have already been taken far enough.

The main conclusion here is that the combined requirements of strings, lute acoustics, stability, and limits on plucking will determine fully the design and operation of a lute. Forcing by the strings is contained within these relationships, but with scope for a criterion that may give a value for an optimum soundboard thickness. From the analysis it can also be seen that for a required pitch of lute, and hence length of string with preferred properties, the working tension would be all that needs to be specified for the best balance of acoustics and stability in a fine design. This in turn shows that the only way of increasing the loudness would be an increase in the string tensions, but with a loss of the best resonances and stability. This indicates that a lute might not be made louder without losing its famous beauty of tone and expression. Interesting comparisons are a simple bow harp with no limit on plucking. For a harpsichord the acoustics and a fixed degree of plucking are related, but the stability is not involved strongly, while for a clavichord the amount of striking can be graded. All these have wider choices of their numerous strings. The lute, maybe uniquely, has a complete set of interrelations between all the processes.

8. Practical variations with size

Several new fundamental relations between strings, acoustics and stability have been explored over a very wide range of possible designs. This provides interesting general understanding, and may be useful. However, from an immediately practical angle it is known that any given lute may sound best with certain string gauges and tensions, and that seemingly small changes can make a large difference. Attempts are even made to alter the pitch of a lute by changing strings and tensions. It was mentioned in section 2 that different lutes for use at a certain pitch may have a range of string lengths, with a variation of about 15%. These effects can be examined with some practical examples. This will show how well the theory can begin to explain smaller details. For some remaining difficult areas further ideas can be suggested.

(i) The first extreme example is a typical G lute, which may actually be adapted for use as a D lute. If a seven course G lute has a bass D string, then the lowest six courses can provide all the notes of a D lute. The only alteration needed is tuning down, or changing, the fourth course f string to an e. One could go further and move up all the strings, and even add a deep bass A. While this has all the correct notes for a much larger D lute, a main problem is a distinct lack of the brilliant tone from a thinner top string. For gut strings the thicker lower strings would also present a large problem. While this is not so severe with metal-wound strings, but more are needed, and the overall feel would be heavy. A further problem is that the spectrum of resonances for the body of a good G lute would have been adapted by historical example and modern experience to a higher range of frequencies suitable for the G lute. A satisfactory aspect of this large alteration is that the tensions can be kept constant, which means that stability and forcing by the strings are not disturbed.

(ii) A similar milder case might be a G lute adapted to F pitch. Again with a change of strings, the bright upper strings are degraded and the resonances may be misplaced. Although the change seems small, the practical results can be very sensitive. While the analyses help understanding, there are two fundamental problems.
The principal problem is very general and concerns the necessary or best position and strength of resonances for a given range of notes. The present work has tried to predict resonances of lutes from their dimensions and material properties. It might be expected that if there were firm values over the whole practical range, then the question of positioning could be solved. This could also lead to further acoustically preferred dimensions of a lute and the bars for a given pitch or size. However, decades of testing violins and guitars may have provided no answer apart from many records for a single size of instrument, with the larger viola and cello receiving little attention. This problem is far deeper and more elusive than having no idea what notes are produced on a wind instrument. Possible approaches are suggested in section 9.

A more specific question concerns the combinations of string tension and diameter that produce brightness. Before the present work this effect, together with thin bass strings, was the only connection between different lute sizes. Originally it was based on a small practical region where gut needed to be taken as far as failure. However, no intrinsic dependence on a breaking point is expected, and equivalent synthetic strings operate far below failure. The problems of low elasticity, inharmonicity, damping, etc for lower strings will become markedly, but gradually, smaller for higher strings. Although a top string is highly stretched its vibration may still be sensitive to elastic stretching and non-linearity. Lute strings are much thinner, longer and at lower tensions than many other instruments, and still a little floppy. The final brightness of slender upper strings with many precise higher harmonics may depend on reducing further all sources of frequency variation. More discussion and a possible solution are given in (viii).

(iii) Another practical way of tuning a G lute lower might be to reduce the string tension. For a pitch in D, the reduction would be a factor of 0.56 (or (3/4)^2 from paper 2), and for F the factor is 0.79. Even this latter small change in pitch requires a decrease in tension from about 3 to 2.4kg. This could still remove the treble brightness and have misplaced resonances. It would also slacken the touch, causing strings to slap; reduce the forcing and loudness and not make good use of the stability limit.

(iv) The reverse possibility of tuning a lute to a higher pitch shows another aspect of the problem. For the previous pitches, the unlikely change of a bass D lute up to G could increase tension by a factor of 1.8 up to 5.4kg, or use thinner strings. A high tension would compromise the stability and be hard to play. The forcing and loudness would be increased greatly, which could be an advantage. The lute resonances would be unnecessarily low and lack higher frequencies. A thicker soundboard could raise these, but might leave a gap in the mid range, although the higher tension would assist the forcing. This has many of the potential advantages found for a larger lute in the stability analysis. However, the larger string length is inconvenient, and lower pitch versions would not be possible. This approach to making a high A lute from a G, or a G from an F, could be useful. In the second option of reducing string diameter, a factor of 0.75 would reduce a treble to a thin 0.3mm. This would not be possible for the gut on old lutes, and for synthetics this fine diameter could have an odd feel. There would be misplaced resonances and no lower pitch models, but no effect on the stability.

(v) From these examples it can be appreciated that even a tone change in any single effect may be far too large. However, a combination of much smaller changes in the resonances, string diameter and tension could allow a total variation of up to about a tone in basic sounding length. For example, a nominal 60cm G lute could be varied between about 56 and 64cm. This overall estimate is preferable to trying to list all the possibilities. A normal type of variation would be longer, slightly thinner and tauter strings with rich harmonics, and lower resonances of a larger lute. The rarer extreme is a G lute with a well-rounded body, perhaps only five bars, yet a string length as short as 56cm, with slightly thicker slacker strings. Possible mistakes would be excessive mass and stability, or resonances too high or deep. Even fine lutes with good resonances and average string length may acquire incorrect strings. One can marvel at the skill
in designing and making a lute with an optimum use of resonances and stability at a given pitch. Lesser instruments constrained to a given pitch may never get to reveal their best sound.

(vi) The mention of larger more stable lutes being able to use a greater tension has so far referred to only the larger sizes of renaissance lutes and the baroque F lute. Modern versions of the former may have slightly greater average tensions, up to about 3.5kg, with diameters increased by about 8% (or $\sqrt{3.5/3}$). Baroque lutes with many more strings could sometimes need extra strengthening. The different types of long neck extension might not greatly affect the stability of the bodies, but the necks need to be strong and aligned well. The long unfretted bass strings were adopted later to avoid all the aspects of poor sound discussed in paper 2. For a string with twice the basic length, the diameter required for the same pitch is halved, and the problems almost completely removed. The total mass of string is also halved, which may seem surprising. The maximum tension used in these various theorbos, chitarrones, archlutes, etc may be about 4kg per string, with diameters increased by about 15%. There may be as many as ten extra courses added to the basic six, and most instruments had single courses, with no octave strings. Even these extra tensions are not enormous in relation to the predicted allowed increase as $T^2$, and some of these lutes also had relatively large basic string lengths and bodies, which improves stability. As shown in section 7, the forcing and loudness depends on the tension $T_i$ in a string. The amplitudes of vibration in the belly $y$ are proportional to $T_i$, so that the vibration energy increases as $T_i^2$, a possible relation being $(T_i^2/T_0)(pk/2q)$. A relatively modest 4kg is therefore almost twice as good as 3kg, or equivalent to a double course.

(vii) After treating variations of string length and the tensions for large lutes, the detail of tension variations across a single lute can be examined. In modern practice the average 3kg per string is usually increased gradually up to 3.5 or even 4kg for the higher courses, and reduced to about 2.7kg for the basses, with 2.5kg for the octave strings. In the present context this variation is not large, but it requires relatively lighter bass strings and thicker trebles. For example, the simple factor of 4 between the effective diameters of bass and treble strings could be reduced to about 3½ (or $\sqrt{2.7/3.5}$), which is close to modern schemes. However, the feel in playing and the sound would be greatly altered if a constant tension were to be adopted. For baroque violins, the old written instructions that all four strings should have a constant tension of about 7kg, rather than a conventional decrease down to 5½kg, has now been taken seriously (Ref 3). This would require the lower strings to be 13% thicker ( or $\sqrt{7/5.5}$), needing a more rigorous bow action, which produces a stronger sound. A similar result would be found for lower strings on a lute, and Ref 3 also points out that Dowland advised equal feel. This would need relatively thicker gut basses than generally used nowadays, which may increase their practical problems. Metal-wound basses would have a loud sustained tone, needing reduction by the lower tension. An open mind may be required on the true nature of early practice, but a reluctance to change a technique of thirty years is understandable.

(viii) Physical criteria are required for the tone of treble strings, as noted in (ii). The similar problem for soundboards unexpectedly produced a possible explanation in the relative strengths of driving the bars and a soundboard. An elastic plate does not need an applied tension for vibration, and in this one respect may be simpler than a string with a range of possible tensions. Both situations are analogous to a thin or soft reed. Here, the uncontrolled vibration may be restrained partially by embouchure, but not in organs etc. Excessive reediness in a belly or slack string can be reduced by an unnaturally small amplitude of plucking. However, this in turn is discouraged by the low resistance, and would give low forcing, so that better design is the only solution. The usefulness of the elastic modulus criterion for reducing stretch sharpening in plucking, non-linearity and inharmonicity in paper 2, applied mainly to the lower strings. These sources of deviation in the ideal frequencies or ($\Delta f/f$) can be compared for a typical solid material such as nylon or gut, and with treble strings say 0.4mm in diameter. The ($\Delta f/f$), for inharmonicity is of order $10^{-5} \text{n}^2$, from paper 5 and might become significant at high overtones,
towards the tenth. For a middle string twice as thick, \((\Delta f/f)\) is 16 times greater and part of the
need for more elastic materials, which become even more necessary for the basses. The other
larger effect was stretch sharpening, for which \((\Delta f/f)\) can be expressed in forms such as:

\[
(\Delta f/f)_s = \frac{E \rho^2}{(3pT_f)^4} = \frac{(Ed^2/T)(p/q)^2}{16} = \frac{E d^2 F^2}{16 T^3}
\]

With approximate typical values for \(p\), \(E\), \(f\), and \(p\) of 1gm/cc, 3GPa, 392Hz for a treble g, 60cm
and 5mm, \((\Delta f/f)_s\) is about \(10^{-3}\), which is far greater than the inharmonicity. These relations use
a plucking position \(q = \frac{1}{4} l\), and also give a large initial stretch \(x_0\) in tuning of about 40mm. The
variation of frequency is about \(1\frac{1}{2}Hz\) in the third harmonic. This is significant but a finer degree of
accuracy than the requirement of decreasing fret sharpening and improving basic tuning in
the lower strings. For a middle g, \((\Delta f/f)_s\) is \(4 \times 10^{-3}\) and increases strongly as \(1/f^2\) and also \(p^2\),
which indicates how definition and brightness may be lost progressively for second and third
courses. It is also seen that a strict use of the modulus criterion with \(E\) proportional to \(p\).

The second expression indicates how brightness could increase simply with stress \(\sigma\) or \(T/d^2\). The final form
expresses the amount of plucking \(p\) in terms of the force, \(F = pT/q\), applied by a player. This has
a limit, so that \((\Delta f/f)_s\) is reduced strongly by increasing \(T\), which is a measure of resistance or
feel. More could be said about touch, but much is learnt familiarity, as in adapting to a single
second course. The dependence of \((\Delta f/f)_s\) on \(p^2\) is part of the non-linearity, but another aspect is
the generation of a small extra component of frequency \(3f\) with an amplitude \((\Delta f/f)_s p/12\). The
large amplitude odd motions of a mass on a slack string are a simpler way to see these effects.

9. Possible future work

This communication grew from a short version of section 8, in an attempt to provide a rounded
account - and some release from this long preoccupation. Some topics required to fill gaps, or
any renewed interest, can be mentioned.

A coupling together of the individual theories for vibrations of the lute body, the enclosed air,
the lower modes of bars, and the higher frequencies of a soundboard, would be interesting. The
estimates of forcing by strings in section 7 began to include some coupling between bars,
soundboard and bridge. The work on guitars and violins, cited in paper 4, involved comparisons
of experiments with the electrical circuit analogues for coupling a top plate to the air and thence
to a bottom plate. A breathing or bellows action slightly lowers the peak associated with the
Helmholtz resonance, and this is also the mechanism of a bass reflex loudspeaker. For these
instruments the ribs were treated as rigid. In contrast, a lute may have a stronger direct coupling
between the more flexible body shell and the belly, but the deeper volume of air may have a
weaker effect. The frequency of possible body vibrations was estimated in paper 4 as a very low
30 to 70Hz. This would be driven weakly by the distant strings, but might produce some
moderate resonance peaks well below the Helmholtz region. The theory of vibrating shells is
difficult, and experiments would be useful for this important region. Records of complete
spectra of resonances for various sizes of lute, in combination with theory, might begin to reveal
the best relative positions and strengths of resonances for each pitch of lute.

No attempt will be made at a short final set of conclusions for the work in all these
communications. Experiments are clearly necessary on many aspects of physics for lutes. On the
other hand, work based entirely on testing or computing without a theoretical basis may risk
becoming a collection of practical information, but without the end product of a fine instrument.
A preferred approach would have been theory, maybe not taken to the present length, forming a
background to a wide range of simple tests, from which important real features could emerge.

References